# Optimal lower bounds for multiple recurrence

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Multiple recurrence concerns the study of the largeness of sets of the form

$$\{n \in \mathbb{N} \colon \mu(A \cap T^{f_1(n)}A \cap T^{f_2(n)}A \cap \cdots \cap T^{f_k(n)}A) > C\mu(A)^{k+1}\} \tag{1}$$

where $A$ is a measurable set of the invertible measure preserving system $(X, \mathcal{B}, \mu, T)$, $C > 0$. and $(f_1, \ldots, f_k)$ are functions $f_i : \mathbb{N} \to \mathbb{Z}$. In this talk I will present recent work on this problem for different functions $f_i$. For instance, if $k \leq 3$ and $f_i(n) = if(n), 1 \leq i \leq k$, we show that (1) has positive density when $f$ is a polynomial along primes with $f(1) = 0$, or a Hardy field function away from polynomials, and (1) is syndetic when $f$ is a Beatty sequence. For $f_i(n) = a_i n, 1 \leq i \leq k$, where $a_i$ are distinct integers, we show that (1) can be empty for $k \geq 4$, and that the largeness of (1) is equivalent to a solution counting problem for certain linear equations when $k = 3$. We also provide partial results on the largeness of (1) when $f_i, 1 \leq i \leq k$ are polynomials. This is joint work with Ahn Le, Joel Moreira and Wenbo Sun.