DERIVATION OF THE ADJOINT OF A WIDE ANGLE PARABOLIC EQUATION FOR ACOUSTIC INVERSION

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Recently the concept of optimal control by adjoint modelling has been introduced in shallow water acoustics for solving inverse problems. A continuous analytic adjoint model has been derived for the standard parabolic equation (SPE) using both local and non local impedance boundary conditions at the water-sediment interface. In this paper we discuss the extension of this approach to the Claerbout wide-angle approximation (WAPE) for these two different boundary conditions. By applying an “equivalent medium” approach we develop the WAPE adjoint and present example inversion results based on synthetic acoustic fields. Further examples are given in an accompanying paper which focusses on the application of penalization methods to the adjoint optimization formalism. The overall performance of the resulting inversion procedure seems promising especially with regard to a rapid environmental assessment (REA) system, where it could for example provide a good initial solution.

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1. INTRODUCTION

Originally, the mathematical concept of optimal control theory was formulated in [1] back in 1971. Adjoint models are nowadays being used for data assimilation, model tuning and sensitivity analysis in several fields, among which meteorology, oceanography and seismics are the most common. The adjoint approach is also being applied as optimal design method in computational fluid dynamics, particularly for aeronautical applications, and for shape reconstruction in inverse scattering.
Currently there is some ongoing research for applying the concept of optimal control by means of an adjoint model to inverse problems in ocean acoustics [2]-[5].

Basically the idea is the same, while [2]-[4] use a continuous and discrete adjoint of a standard parabolic approximation (SPE) of the Helmholtz equation respectively, [5] applies the adjoint of a normal mode model in order to compute the gradient of a cost function with respect to one or more control variables. When implemented in an effective gradient-based minimization algorithm the number of modelling runs required for the inversion of an acoustic field can thus be significantly decreased.

In the following we extend the analytic adjoint SPE approach in [2],[3] to the Claerbout wide-angle approximation (WAPE) for two types of boundary conditions at the water-sediment interface. By applying an “equivalent medium” approach we develop the WAPE adjoint and present example inversion results based on synthetic acoustic fields. In a companion paper [8] the application of penalization methods to the adjoint optimization formalism is investigated and further examples are given.

2. ADJOINT FORMALISM

If we consider a dynamical physical system and a model $G$ describing this system and suppose that the model can compute a solution $\nu$ corresponding to a set of observations, we introduce, without loss of generality, a least-squares cost function $J$ in order to quantify the model data fit at a given range. With $E, F$ Hilbert spaces

$$J : F \rightarrow \mathbb{R}$$

$$\nu \mapsto J(\nu)$$

If we further specify a set of control variables $\chi$ in order to manipulate the model

$$G : E \rightarrow F$$

$$\chi \mapsto \nu = G(\chi)$$

the gradient of the cost function with respect to these control variables can be evaluated using the definition of the adjoint operator as follows

$$\nabla_\chi J = G^* \nabla_\nu J$$

The linear operator $G^*$ represents the so-called tangent linear model (TLM) and its adjoint $G^{**}$ which is linear as well, represents the adjoint model. $\nabla_\nu J$ is the residual norm of the measured-predicted acoustic field which can be interpreted as a forcing of the adjoint model.

3. OPTIMAL BOUNDARY CONTROL PROBLEM

According to the adjoint formalism presented in the previous section, the chosen numerical model $G$ is the wide angle parabolic equation (WAPE) due to Claerbout, which is obtained by introducing a rational Padé-[1,1] series expansion of the square root operator in the Helmholtz equation. In this paper the above mentioned control variables are determined by the bottom boundary condition at the water-sediment interface, which categorizes the presented approach as an optimal boundary control method for the wide angle parabolic equation.

To summarize, the system of the wide angle parabolic model including a Dirichlet boundary condition at the surface, an analytical Greene’s source term and both a simple Robin boundary condition (LBC) and a generalized non-local boundary condition (NLBC) based on [7] at the
bottom can be described as follows

\[
\begin{cases}
2ik_0 \left[ 1 + \frac{1}{4}(n^2 - 1) \right] \psi_r + \psi_{zz} + k_0^2(n^2 - 1) \psi + \frac{i}{2k_0} \psi_{zz,r} = 0 \\
\psi(r, z = 0) = 0 \\
p(r, z = 0) = 0 \\
\psi(r = 0, z) = S(z) \\
\psi_z(r) + i\gamma(r) \psi(r) \bigg|_{z=H} = 0 \\
\psi_z(r) - i\beta \psi(r) \bigg|_{z=H} = F(r)
\end{cases}
\]

4. DERIVATION OF THE ADJOINT MODEL

With the cost function measuring the fit between the observed \( \psi_{\text{obs}} \) and predicted \( \psi \) pressure fields at the maximum range \( r = R \)

\[
J(\chi) = \frac{1}{2} \int_{z=0}^{H} |\psi(\chi, R, z) - \psi_{\text{obs}}(R, z)|^2 dz
\]

and the choice of an appropriate inner product \( \langle \cdot, \cdot \rangle \) the aim is to use the TLM derived in Sec. 3 in order to evaluate \( J'(\chi, \phi) \) and finally get an expression for \( \nabla J(\phi) \):

\[
J'(\chi, \phi) = \int_{z,r=R} \langle \psi - \psi_{\text{obs}}, w \rangle \, dz = \int_{z,r=R} \langle \nabla J(\phi), \phi \rangle
\]

Calculating the scalar product of the wide angle TLM PE and the newly introduced adjoint variable \( p \) gives the following integral

\[
\int \int_{z,r} \left\langle 2ik_0 \left[ 1 + \frac{1}{4}(n^2 - 1) \right] \, w_r + w_{zz} + k_0^2(n^2 - 1)w \right. \\
\left. + \frac{i}{2k_0}w_{zz,r} \right, p \rangle \, drdz = 0
\]

By partial integration and repeated use of Stoke’s theorem, appropriate conditions for the adjoint variable \( p \) have to be identified in order to simplify the integral equation such that the remainder can be expressed in a form similar to Eq. (7).

Leaving out details of the integration by parts, we define in a first step

\[
\begin{cases}
2ik_0 \left[ 1 + \frac{1}{4}(n^2 - 1) \right] p_r + p_{zz} + k_0^2(n^2 - 1)p + \frac{i}{2k_0}p_{zz,r} = 0 \\
p(r, z = 0) = 0 \\
-2ik_0 \left[ 1 + \frac{1}{4}(n^2 - 1) \right] p(z) - \frac{i}{2k_0} p_{zz}(z) \bigg|_{r=R} = \psi(z) - \psi_{\text{obs}}(z) \bigg|_{r=R}
\end{cases}
\]
as part of the WAPE adjoint model, and sort the remaining terms from Eq. (8) by integration region to finally obtain

\[
\int_{z,r=R} \langle w, \psi - \psi_{obs} \rangle + \int_{r,z=H} \langle w_z, p \rangle - \langle w, p_z \rangle \, dr + \ldots + \frac{1}{2k_0} \int_{r,z=H} -\langle iw_z, p_z \rangle + \langle iw, p_{r,z} \rangle + \ldots + \frac{1}{2k_0} \left( \langle iw_z, p \rangle \right)_{z=H,r=R} - \langle iw, p_z \rangle \right)_{z=H,r=R} = 0
\]

(10)

As a second step, insertion of the boundary condition \( w_z = -i\varphi \psi - i\gamma w \) yields the following expression for the missing LBC adjoint boundary condition at \( z = H \)

\[
\left[ -i\gamma(r) + \frac{\partial}{\partial z} \left( p(r) + \frac{i}{2k_0} p_r(r) \right) \right]_{z=H} = 0
\]

(11)

and the corresponding gradient of the cost function

\[
\nabla J = -i\bar{\psi}(r) \left( p(r) + \frac{i}{2k_0} p_r(r) \right) \right|_{z=H}
\]

(12)

In the NLBC case with the controls \([\beta, F]\), insertion of the respective TLM boundary condition \( w_z = f + i\beta w + ib\psi \) completes the adjoint WAPE system with

\[
\left[ 1 + \frac{i}{2k_0} \frac{\partial}{\partial r} \right] \left( p_z(r) + i\beta p(r) \right) \right|_{z=H} = 0
\]

(13)

and leads to the following formula for the gradient of the cost function

\[
\nabla J = \left[ \frac{i}{\bar{\psi}(r)} \left( p(r) + \frac{i}{2k_0} p_r(r) \right) \right] - \left( p(r) + \frac{i}{2k_0} p_r(r) \right)
\]

(14)

The resulting WAPE adjoint equation which governs the behaviour of the adjoint field \( p \) is equivalent to the direct model but backwards in space and contains a complex transparent boundary condition at \( z = H \) and an implicit adjoint initial condition at range \( r = R \). In contrast to [2]–[5] where the adjoint field was created by using the phase-conjugated (time-reversed) error residual at each receiver as an acoustic source, this interpretation is not valid for the adjoint WAPE anymore. The Neumann adjoint starter condition at \( r = R \) in Eq. (9) replaces the simple adjoint Dirichlet condition at \( r = R \) in the former cases.

5. Numerical Implementation

The finite difference implementation of the direct problem is a standard Crank-Nicholson scheme whereas the discrete formulation of the adjoint system including transparent boundary condition and implicit adjoint starter condition requires a separate treatment of the first range step. Starting from range \( r = R \) we therefore apply an implicit backward-centred (BW-CT) Euler method and only afterwards proceed with the standard Crank-Nicholson scheme. In regard to the minimization algorithm, the adjoint model is embedded in a conjugate-gradient optimization method to determine the optimal controls.

Figs. 1 and 2 show an example of inversion results for a frequency of 500 Hz generated by a Greene’s source in a water column with constant sound speed of 1500 m/s. The true non local impedance boundary conditions were calculated for a sandy reflecting bottom with a compression speed of 1600 m/s, an attenuation of 0.5 dB/\( \lambda \) and a density of 1.8 g/cm\(^3\), while the initial ones were obtained with a soft sediment characterized by a low compression speed of 1505 m/s. Although the initial field in Fig. 1 was quite different from the true field, the reconstructed field agrees very well with the true one. Also the control parameter \( F \) in Fig. 2 is well retrieved upon completion of the inversion process.
Figure 1: Amplitude of the initial (a), true (b) and inverted (c) acoustic fields for a frequency of 500 Hz. After the assimilation process, the inverted and the true fields are nearly identical.

Figure 2: (a) Convergence of the algorithm vs. iteration number; (b) Relative error between the initial and true pressure fields (green), and between the calculated and true fields (blue); (c) Comparison of the initial (red), true (green) and inverted (blue) control parameter $F$ vs. range; (d) Same as (c) for the field vs. depth at maximum range. Only the imaginary parts of $F$ and $\psi$ are shown. The real parts behave similarly.
6. CONCLUSION

The continuous analytical adjoint approach, initially developed for the standard parabolic equation [2],[3], was extended to the physically realistic model of a wide angle parabolic equation. The adjoint derivation was implemented numerically and successfully tested with synthetic acoustic fields. Preliminary results indicate that the wide-angle adjoint is accurate and computationally very efficient: 16 iterations are sufficient to recover the field with an overall error less than 5%. Work is ongoing to regularize and test the adjoint approach on real-world benchmarks with the aim to invert for acoustic properties of the ocean bottom.

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REFERENCES