Triangulations équivariantes des tores des groupes de Lie et tores hyperboliques pour les groupes de Coxeter non cristallographiques

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$R\Gamma(X, \underline{\mathbb{Z}}) \in \mathcal{D}^b(\mathbb{Z}[W])$ and equivariant cellular structures

discrete group $W \circlearrowright X$ topological space

$$\rightsquigarrow \quad W \circlearrowright H^*(X, \mathbb{Z}) = H^*(R\Gamma(X, \underline{\mathbb{Z}})).$$

Also, $R\Gamma(X, \underline{\mathbb{Z}}) \in \mathcal{D}^b(\mathbb{Z}[W])$, but how to compute $R\Gamma(X, \underline{\mathbb{Z}})$?

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Definition

A CW-structure on X is W-**equivariant** if

- W acts on cells
- For $e \subset X$ a cell and $w \in W$, if $we = e$ then $w_{|e} = \mathrm{id}_e$.

Associated **cellular chain complex**: $C_*^{\mathrm{cell}}(X, W; \mathbb{Z}) \in \mathcal{C}_b(\mathbb{Z}[W])$.

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Theorem

The complex $C^*_{\mathrm{cell}}(X, W; \mathbb{Z})$ is well-defined up to homotopy and $C^*_{\mathrm{cell}}(X, W; \mathbb{Z}) \cong R\Gamma(X,\underline{\mathbb{Z}})$ in $\mathcal{D}^b(\mathbb{Z}[W])$.

Illustration: $\{\pm 1\} \circlearrowright \mathbb{S}^2 \subset \mathbb{R}^3$

$C_2 = \{1, s\}$ acts on $\mathbb{S}^2$ via the antipode $s : x \mapsto -x$. We construct a C_2-equivariant cellular structure as follows:

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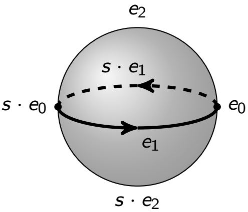

Chain complex given by

$$C_*^{\mathrm{cell}}(\mathbb{S}^2, C_2; \mathbb{Z}) = \left(\mathbb{Z}[C_2] \langle e_2 \rangle \xrightarrow{1+s} \mathbb{Z}[C_2] \langle e_1 \rangle \xrightarrow{1-s} \mathbb{Z}[C_2] \langle e_0 \rangle \right)$$

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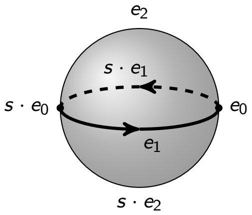

Cochain complex

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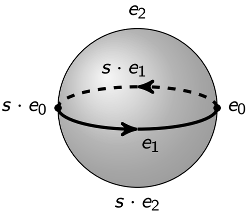

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so $R\Gamma(\mathbb{S}^2, \underline{\mathbb{Q}}) \simeq \mathbb{1} \oplus \varepsilon[-2]$ and $R\Gamma(\mathbb{S}^2, \underline{\mathbb{F}_2})$ has cohomology $H^*(\mathbb{S}^2, \mathbb{F}_2) = \mathbb{1} \oplus \mathbb{1}[-2]$, however, $R\Gamma(\mathbb{S}^2, \underline{\mathbb{F}_2})$ is indecomposable...

Position of the problem

Notation:

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Problem (A)

Exhibit a W-equivariant triangulation of T.

- **Natural method**: Describe T as a W-equivariant simplicial complex.
- **Reduction**: use the exponential $\mathfrak{t} := \mathrm{Lie}(T) \twoheadrightarrow T$ and work with $\mathfrak{t}$.

Position of the problem

Notation:

- $\Phi \subset i\mathfrak{t}^* =: V$ *root system* of (G, T) (Bourbaki), $\Phi^\vee \subset V^*$,
- $\Pi \subset \Phi^+ \subset \Phi$ positive and simple roots, with $\Pi \approx \{1, \ldots, n\}$,
- $Q := \mathbb{Z}\Phi$ (resp. $Q^\vee := \mathbb{Z}\Phi^\vee$) (co)root lattice,
- P (resp. $P^\vee$) (co)weight lattice,
- Finally, $X(T) := \{d\lambda : \mathfrak{t} \to i\mathbb{R} \ ; \ \lambda \in \mathrm{Hom}(T, \mathbb{S}^1)\} \subset V$ character lattice of T and $Y(T) := X(T)^\wedge$ cocharacters.

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There is an isomorphism W-Lie groups:

$$\exp : V^*/Y(T) \xrightarrow{\sim} T.$$

We also have

$$P/X(T) \simeq \pi_1(G).$$

(Extended) affine Weyl groups

We work with an irreducible **root datum** $R := (X, \Phi, Y, \Phi^\vee)$ with Weyl group W and $V := \mathbb{R} \otimes \mathbb{Z}\Phi$.

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Example: the type A_{n-1}:

	Simply conneted	In between	Adjoint
G	$SU_n(\mathbb{C})$		
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W	$\mathfrak{S}_n$		
$X := X(T)$	P		
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W	$\mathfrak{S}_n$	$\mathfrak{S}_n$	$\mathfrak{S}_n$
$X := X(T)$	P	$Q \subset X \subset P$; $[P : X] = d$	Q
$Y := Y(T)$	$Q^\vee$	$P^\vee \supset Y \supset Q^\vee$	$P^\vee$
$\pi_1(G) = P/X$	1	$\mathbb{Z}/d\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$

The simply-connected case

If $\pi_1(G) = 1$ then $Y = Q^\vee$ and $Q^\vee \rtimes W \simeq W_{\mathrm{a}}$ is the **affine Weyl group**, a Coxeter group.

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For $\alpha_i \in \Pi$ let $\widehat{s}_i := s_{\alpha_i}$ and for $\alpha_0 \in \Phi^+$ the **highest root**, $\widehat{s}_0 := s_{\alpha_0} + \alpha_0^\vee$, then

$$W_{\mathrm{a}} \simeq \left\langle \widehat{s}_0, \widehat{s}_1, \ldots, \widehat{s}_n \;\middle|\; \forall 0 \leq i, j \leq n,\ (\widehat{s}_i \widehat{s}_j)^{o(s_{\alpha_i} s_{\alpha_j})} = 1 \right\rangle .$$

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Fundamental domain for $W_{\mathrm{a}} \circlearrowright V^*$? **the fundamental alcove**:

$$\mathcal{A} := \{\lambda \in V^* \ ;\ \forall \alpha \in \Phi^+,\ 0 \leq \lambda(\alpha) \leq 1\} \simeq \Delta^n.$$

The associated chain complex

Theorem (G. 2020)

The face lattice of $\mathcal{A} \simeq \Delta^n$ induces a W_{a}-triangulation of V^* whose associated cellular complex $C_*^{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z})$ is

$$\cdots \longrightarrow \bigoplus_{|I|=n-k} \mathbb{Z}[W_{\mathrm{a}}^I] \xrightarrow{\partial_k} \bigoplus_{|I|=n-k+1} \mathbb{Z}[W_{\mathrm{a}}^I] \longrightarrow \cdots$$

where $W_{\mathrm{a}}^I := \{w \in W_{\mathrm{a}} \ ; \ \ell(ws_i) > \ell(w), \ \forall i \in I\} \approx W_{\mathrm{a}}/(W_{\mathrm{a}})_I$ is the set of minimal length coset representatives and

$$\{0, \ldots, n\} \backslash I = \{i_1 < \ldots < i_{k+1}\} \Rightarrow (\partial_k)_{|\mathbb{Z}[W_{\mathrm{a}}^I]} = \sum_{u=1}^{k+1} (-1)^u p^I_{I \cup \{i_u\}},$$

$$p^I_J : W_{\mathrm{a}}^I \twoheadrightarrow W_{\mathrm{a}}^J \quad \text{for} \quad I \subset J.$$

Moreover, the cup product $\mathbb{Z}[{}^I W_{\mathrm{a}}] \otimes \mathbb{Z}[{}^J W_{\mathrm{a}}] \xrightarrow{\cup} \mathbb{Z}[{}^{I \cap J} W_{\mathrm{a}}]$ giving the $\mathbb{Z}[W_{\mathrm{a}}]$-dg-ring structure of the complex is explicitly computed.

Consequences for T

Corollary (A1)

The $\mathbb{Z}[W]$*-dg-ring* $C^*_{\mathrm{cell}}(V^*/Q^\vee, W; \mathbb{Z})$ *is given by*

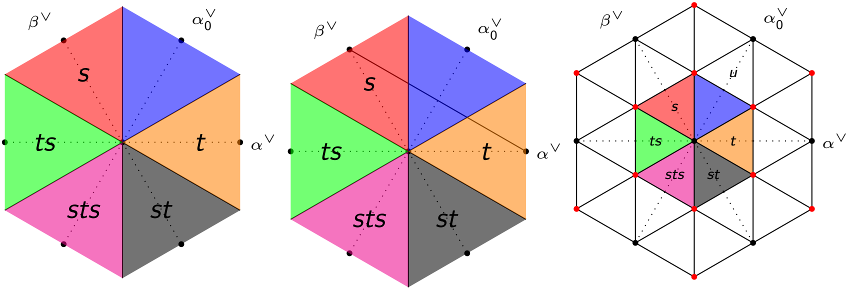
$$C^*_{\mathrm{cell}}(V^*/Q^\vee, W; \mathbb{Z}) = \mathrm{Def}_W^{W_{\mathrm{a}}}(C^*_{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z})),$$

with $\mathrm{Def}_W^{W_{\mathrm{a}}} : \mathbb{Z}[W_{\mathrm{a}}]-\mathbf{dgRing} \longrightarrow \mathbb{Z}[W]-\mathbf{dgRing}$ *the functor induced by the deflation.*
Abusing the notation, $W_I := \langle s_{\alpha_i},\ i \in I \rangle \leq W$*, we have*

$$\forall k \geq 0,\ C^k_{\mathrm{cell}}(V^*/Q^\vee, W; \mathbb{Z}) = \bigoplus_{I \subset \{0,\ldots,n\}\ ;\ |I| = n-k} \mathbb{Z}[W_I \backslash W].$$

The cohomology algebra is $H^\bullet(V^*/Q^\vee, \mathbb{Z}) \simeq \Lambda^\bullet_{\mathbb{Z}}(P)$.

Example in type A_2



(a) Fundamental chamber (in blue) and its $\mathfrak{S}_3$-translates.

(b) What if we add a wall?

(c) Alcoves for $(\mathfrak{S}_3)_{\mathrm{a}} = \langle s, t, u \rangle$.

Figure: Chambers subdivided into alcoves.

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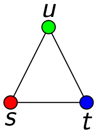
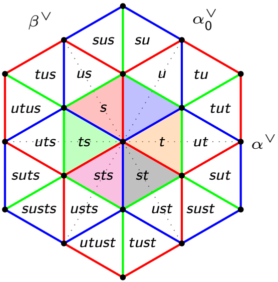
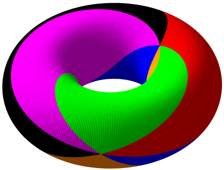


(a) Dynkin diagram of $\widetilde{A}_2$.



(b) Fundamental alcove and some of its $(\mathfrak{S}_3)_{\mathrm{a}}$-translates.



(c) Resulting $\mathfrak{S}_3$-triangulation of $S(U(1)^3) \simeq (\mathbb{S}^1)^2$.

Figure: Triangulation of the torus $S(U(1)^3)$ of $SU(3)$.

The complex $C_*^{\mathrm{cell}}(S(U(1)^3), \mathfrak{S}_3; \mathbb{Z})$ is given by

$$\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}} \mathbb{Z}[\mathfrak{S}_3/\langle s_\beta \rangle] \oplus \mathbb{Z}[\mathfrak{S}_3/\langle s_\alpha s_\beta s_\alpha \rangle] \oplus \mathbb{Z}[\mathfrak{S}_3/\langle s_\alpha \rangle] \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}} \mathbb{Z}^3.$$

General case: barycentric subdivision of $\mathcal{A}$

Problem: the group Ω acts non-trivially on $\mathcal{A}$. However, we have the following comfortable result:

Lemma

*Let Γ be a discrete affine group acting on a polytope Δ. Then the **barycentric subdivision** $\mathrm{Sd}(\Delta)$ is a Γ-triangulation of Δ.*

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Applying this to $\Delta = \mathcal{A}$ and $\Gamma = \Omega$ gives the

Theorem (A2)

The barycentric subdivision of the fundamental alcove induces a $\widehat{W_{\mathrm{a}}}$-equivariant triangulation of $\mathfrak{t}$. The same holds for any W-lattice $Q^\vee \subset \Lambda \subset P^\vee$ and the intermediate group W_Λ.

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We can compute differentials and cup-product, but the formulas are not very enlightening. However, they are implemented in GAP.

Example of A_2

The homology chain complex in the case of $SU(3)$ is

$$\mathbb{Z}[W] \xrightarrow{(1\ 1\ -1)} \mathbb{Z}[W/\langle s_\beta \rangle] \oplus \mathbb{Z}[W/\langle s_\alpha s_\beta s_\alpha \rangle] \oplus \mathbb{Z}[W/\langle s_\alpha \rangle] \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}} \mathbb{Z}^3 .$$

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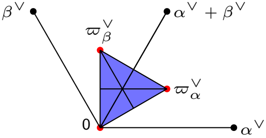

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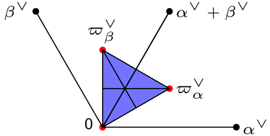

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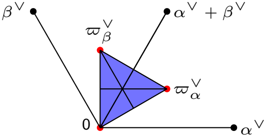

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We have $\Omega = \{1, \omega_\alpha, \omega_\beta\} \simeq \mathbb{Z}/3\mathbb{Z}$, where ω_β the rotation with center $\mathrm{bar}(\mathcal{A})$ and angle $2\pi/3$. The complex for $PSU(3)$ is

$$\mathbb{Z}[W]^2 \xrightarrow{\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & s_\beta s_\alpha \end{pmatrix}} \mathbb{Z}[W/\langle s_\beta \rangle] \oplus \mathbb{Z}[W/\langle s_\alpha \rangle] \oplus \mathbb{Z}[W]^2 \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ -1 & s_\beta s_\alpha & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}[W/\langle s_\beta \rangle] \oplus \mathbb{Z}[W/\langle s_\alpha s_\beta \rangle] \ .$$

Adjoint case: (some) symmetries of the alcove $\mathcal{A}$

Assume now $X = Q$ (i.e. $Y = P^\vee$). Let $\widehat{W_{\mathrm{a}}} := P^\vee \rtimes W$ be the **extended affine Weyl group** and

$$\Omega := \{\widehat{w} \in \widehat{W_{\mathrm{a}}} \ ; \ \widehat{w}(\mathcal{A}) = \mathcal{A}\} \simeq \widehat{W_{\mathrm{a}}}/W_{\mathrm{a}} \simeq P/Q \simeq P^\vee/Q^\vee.$$

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$\Omega \setminus \{1\}$ parametrized by the values at 0: the *minuscule* weights, i.e. those ϖ_i such that $n_i = 1$ in $\alpha_0 = \sum_i n_i \alpha_i$.

Problem: $\widehat{W_{\mathrm{a}}}$ is not Coxeter.

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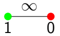
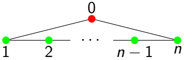
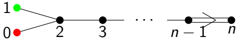
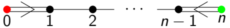
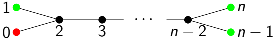
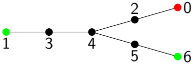
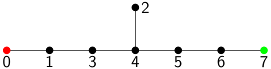
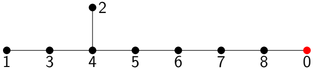
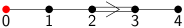
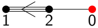
Still possible to find a fundamental polytope $F_\mathrm{ad} \subset \mathcal{A}$ for Ω, using the permutation action of Ω on the affine Dynkin diagram and on minuscule weights.

Type	Affine Dynkin diagram $\mathcal{D}_{\mathrm{a}}$	Fundamental group $\Omega \simeq P/Q$
$\widetilde{A}_1$		$\mathbb{Z}/2\mathbb{Z}$
$\widetilde{A}_n$ $(n \geq 2)$		$\mathbb{Z}/(n+1)\mathbb{Z}$
$\widetilde{B}_n$ $(n \geq 3)$		$\mathbb{Z}/2\mathbb{Z}$
$\widetilde{C}_n$ $(n \geq 2)$		$\mathbb{Z}/2\mathbb{Z}$
$\widetilde{D}_n$ $(n \geq 4)$		$\begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ even} \\ \mathbb{Z}/4\mathbb{Z} & \text{if } n \text{ is odd} \end{cases}$
$\widetilde{E}_6$		$\mathbb{Z}/3\mathbb{Z}$
$\widetilde{E}_7$		$\mathbb{Z}/2\mathbb{Z}$
$\widetilde{E}_8$		1
$\widetilde{F}_4$		1
$\widetilde{G}_2$		1

The Komrakov–Premet polytope

Theorem (Komrakov–Premet 1984, G. 2020)

The convex polytope of V^* defined by

$$F_{\mathrm{ad}} := \{\lambda \in \mathcal{A} \mid \forall \alpha \in \Pi,\ n_\alpha = 1 \ \Rightarrow\ \lambda(\alpha_0 + \alpha) \leq 1\}$$

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Corollary

If G is of adjoint type, then $\exp(F_{\mathrm{ad}}) \simeq F_{\mathrm{ad}}$ is a fundamental domain for W acting on T.

Vertices of F_{ad} and examples

$$\mathcal{A} = \{\lambda \in V^* \mid \forall \alpha \in \Pi,\ \lambda(\alpha) \geq 0,\ \lambda(\alpha_0) \leq 1\} = \mathrm{conv}(\{0\} \cup \{\varpi_i^\vee / n_i\}_i)$$

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$$\begin{aligned} F_{\mathrm{ad}} &= \{\lambda \in \mathcal{A} \mid \forall \alpha \in \Pi,\ n_\alpha = 1 \ \Rightarrow\ \lambda(\alpha_0 + \alpha) \leq 1\} \\ &= \mathrm{conv}\left(\left\{ \frac{\varpi_i^\vee}{n_i} \right\}_{n_i > 1} \cup \left\{ \frac{1}{|J'| + 1} \sum_{j \in J'} \varpi_j^\vee,\ \emptyset \subseteq J' \subseteq J \right\} \right), \end{aligned}$$

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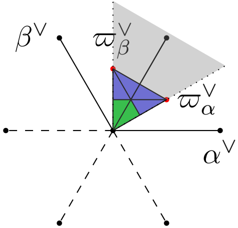
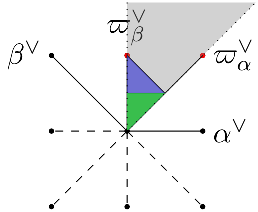
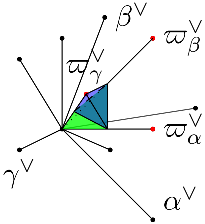


(a) Type A_2.



(b) Type $B_2 = C_2$.



(c) Type C_3.

Intermediate cases?

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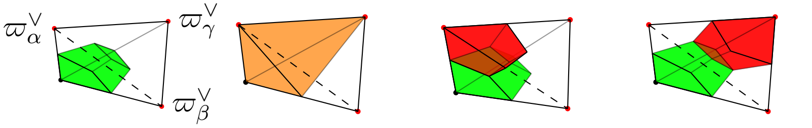

(a) $F_{\mathrm{ad}} \subset \mathcal{A}$ (b) $F_Y \subset \mathcal{A}$ (c) $F_{\mathrm{ad}} \cup \omega_\alpha F_{\mathrm{ad}}$ (d) $F_{\mathrm{ad}} \cup \omega_\gamma F_{\mathrm{ad}}$

Figure: Type A_3 with $\Omega_Y = 2\mathbb{Z}/4\mathbb{Z} \leq \mathbb{Z}/4\mathbb{Z} = \Omega$

Further decompose: affine isometries of an alcove

Let $(V,(\cdot,\cdot))$ be Euclidean (of type A, D or E), with

$$\mathcal{A} = \{x \in V \mid \forall \alpha \in \Phi^+,\ 0 \leq (x,\alpha) \leq 1\}.$$

Fundamental polytope for $\mathrm{Aut}(\mathcal{A}) \circlearrowright \mathcal{A}$?

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Theorem (Seco–G.–Neeb 2025)

Projection to linear part yields an isomorphism

$$\pi : \mathrm{Aut}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Aut}(\mathcal{D}_{\mathrm{a}}).$$

In particular, $\mathrm{Aut}(\mathcal{A}) \simeq \Omega \rtimes \mathrm{Aut}(\mathcal{D})$ *is an abstract Coxeter group.*

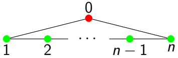
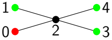
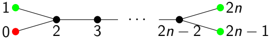
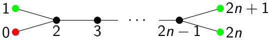
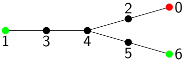
Type	Affine Dynkin diagram $\mathcal{D}_{\mathrm{a}}$	Ω	$\mathrm{Aut}(\mathcal{D})$	$\mathrm{Aut}(\mathcal{D}_{\mathrm{a}}) \simeq \mathrm{Aut}(\mathcal{A})$
$\widetilde{A_n}$ $(n \geq 2)$		$\mathbb{Z}/(n+1)\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$I_2(n+1)$
$\widetilde{D_4}$		$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathfrak{S}_3$	A_3
$\widetilde{D_{2n}}$ $(n > 2)$		$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	B_2
$\widetilde{D_{2n+1}}$ $(n \geq 2)$		$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	B_2
$\widetilde{E_6}$		$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	A_2

Table: Affine Dynkin diagrams with $\mathrm{Aut}(\mathcal{D}) \neq 1$ (i.e. $\Omega \lneq \mathrm{Aut}(\mathcal{D}_{\mathrm{a}})$).

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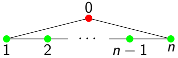
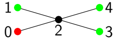
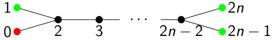
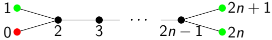
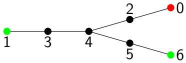
$\widetilde{A_n}$ $(n \geq 2)$		$\mathbb{Z}/(n+1)\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$I_2(n+1)$
$\widetilde{D_4}$		$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathfrak{S}_3$	A_3
$\widetilde{D_{2n}}$ $(n > 2)$		$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	B_2
$\widetilde{D_{2n+1}}$ $(n \geq 2)$		$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	B_2
$\widetilde{E_6}$		$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	A_2

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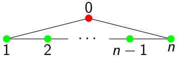
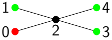
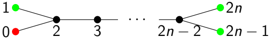
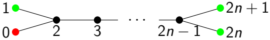
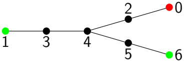
$\widetilde{A_n}$ $(n \geq 2)$		$\mathbb{Z}/(n+1)\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$I_2(n+1)$
$\widetilde{D_4}$		$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathfrak{S}_3$	A_3
$\widetilde{D_{2n}}$ $(n > 2)$		$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	B_2
$\widetilde{D_{2n+1}}$ $(n \geq 2)$		$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	B_2
$\widetilde{E_6}$		$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	A_2

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$\rightsquigarrow \exists f_0, f_1 \in \mathrm{Aut}(\mathcal{D})$ isometric involutions with $\mathrm{Aut}(\mathcal{D}) = \langle f_0, f_1 \rangle$ ($f_0 = f_1$ unless $\Phi = D_4$). Look for a $\langle f_0, f_1 \rangle$-fundamental polytope $F_{\mathrm{iso}} \subset F_{\mathrm{ad}}$.

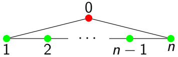
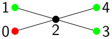
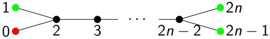
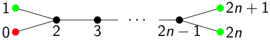
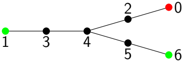
Type	Affine Dynkin diagram $\mathcal{D}_{\mathrm{a}}$	Ω	$\mathrm{Aut}(\mathcal{D})$	$\mathrm{Aut}(\mathcal{D}_{\mathrm{a}}) \simeq \mathrm{Aut}(\mathcal{A})$
$\widetilde{A_n}$ $(n \geq 2)$		$\mathbb{Z}/(n+1)\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$I_2(n+1)$
$\widetilde{D_4}$		$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathfrak{S}_3$	A_3
$\widetilde{D_{2n}}$ $(n > 2)$		$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	B_2
$\widetilde{D_{2n+1}}$ $(n \geq 2)$		$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	B_2
$\widetilde{E_6}$		$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	A_2

Table: Affine Dynkin diagrams with $\mathrm{Aut}(\mathcal{D}) \neq 1$ (i.e. $\Omega \lneq \mathrm{Aut}(\mathcal{D}_{\mathrm{a}})$).

$\rightsquigarrow \exists f_0, f_1 \in \mathrm{Aut}(\mathcal{D})$ isometric involutions with $\mathrm{Aut}(\mathcal{D}) = \langle f_0, f_1 \rangle$ ($f_0 = f_1$ unless $\Phi = D_4$). Look for a $\langle f_0, f_1 \rangle$-fundamental polytope $F_{\mathrm{iso}} \subset F_{\mathrm{ad}}$.

Remark

If $0 \neq v_i \in V$ is such that $f_i(v_i) = -v_i$, then $F' = \{x \in F_{\mathrm{ad}} \mid (x, v_i) \geq 0\}$ is fundamental for $\langle f_i \rangle \circlearrowright F_{\mathrm{ad}} \ldots$

Fundamental polytope F_{iso} for $\mathrm{Aut}(\mathcal{A}) \circlearrowright \mathcal{A}$

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Theorem (Seco–G.–Neeb 2025)

Consider $v_0, v_1 \in V$ as follows

$$\begin{cases} v_0 = v_1 = \sum_{i=1}^{\lfloor n/2 \rfloor} (\alpha_i - \alpha_{n+1-i}) & \text{if } \Phi = A_{n>1}, \\ v_0 = \alpha_1 - \alpha_3, \ v_1 = \alpha_3 - \alpha_4 & \text{if } \Phi = D_4, \\ v_0 = v_1 = \alpha_{n-1} - \alpha_n & \text{if } \Phi = D_{n>4}, \\ v_0 = v_1 = \alpha_1 - \alpha_6 & \text{if } \Phi = E_6, \end{cases}$$

then $F_{\mathrm{iso}} := \{x \in F_{\mathrm{ad}} \mid (x, v_0) \geq 0, \ (x, v_1) \geq 0\}$ is a fundamental polytope for $\mathrm{Aut}(\mathcal{A}) \circlearrowright \mathcal{A}$ and we have

$$\mathrm{vert}(F_{\mathrm{iso}}) = F_{\mathrm{iso}} \cap \mathrm{vert}(F_{\mathrm{ad}}).$$

Examples

Recall $F_{\mathrm{iso}} = \mathrm{conv}(\{x \in \mathrm{vert}(F_{\mathrm{ad}}) \mid (x, v_i) \geq 0\})$

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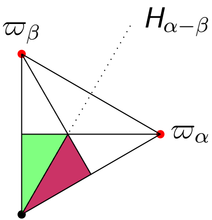
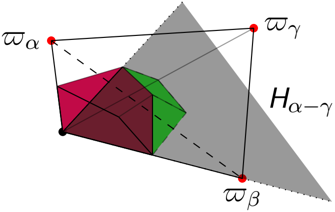


(a) Type A_2.



(b) Type A_3.

Figure: F_{iso}(purple) $\subset F_{\mathrm{ad}}$(green) $\subset \mathcal{A}$

Compact hyperbolic extensions

The combinatorics of the complex for $\pi_1(G) = 1$ makes sense for any Coxeter system (W, S), with an additional reflection $r_W \in W$.

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Geometric interpretation of this analogy?

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Find a reflection giving a "nice" Coxeter extension $(\widehat{W}, S \cup \{\widehat{s}_0\})$?

"True tori": W Weyl, $r_W = s_{\widetilde{\alpha}}$ (highest root), $\widehat{W} = W_{\mathrm{a}}$.

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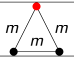
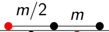
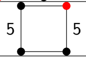
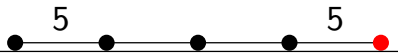
Extension	Coxeter graph
$\widehat{I_2(m)}$ $(m \equiv 1[2])$	
$\widehat{I_2(m)}$ $(m \equiv 0[4])$	
$\widehat{I_2(m)}$ $(m \equiv 2[4])$	
$\widehat{H_3}$	
$\widehat{H_4}$	

The non-commutative lattice Q

If $W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{i,j}} = 1 \rangle$, we let

$$\widehat{W} := \left\langle \widehat{s}_0, \widehat{s}_1, \ldots, \widehat{s}_n \;\middle|\; \forall i,j \geq 1,\; (\widehat{s}_i \widehat{s}_j)^{m_{i,j}} = (\widehat{s}_0 \widehat{s}_i)^{o(r_W s_i)} = \widehat{s}_0^2 = 1 \right\rangle.$$

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Sending $\widehat{s}_0 \in \widehat{W}$ to $r_W \in W$ induces a surjection $\pi : \widehat{W} \twoheadrightarrow W$ and we have $\widehat{W} = Q \rtimes W$, where the torsion-free subgroup

$$Q := \ker(\pi) = \left\langle (\widehat{s}_0 r_W)^{\widehat{W}} \right\rangle \lhd \widehat{W}$$

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Lemma

We have

$$\forall I \subsetneq \{\widehat{s}_0, \ldots, \widehat{s}_n\},\ \widehat{W}_I \cap Q = 1.$$

Equivalently, Q is torsion-free.

Construction of $\mathbf{T}(W)$ from the Coxeter complex

Consider the **Coxeter complex**

$$\Sigma(\widehat{W}) := \left(\bigcup_{w \in \widehat{W}} w(\overline{C} \setminus \{0\}) \right) / \mathbb{R}_+^*,$$

where C is the *fundamental chamber* of $\widehat{W}$. We define

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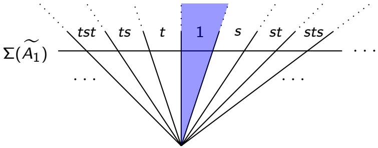
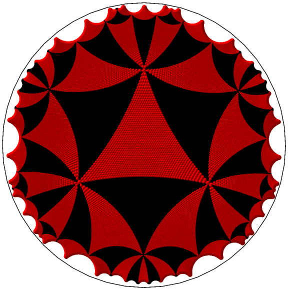


Figure: $\Sigma(\widetilde{A_1}) = \Sigma(I_2(\infty))$ as an affine line.

The example of $I_2(5)$

For $W = I_2(5)$, the simplicial structure of $\Sigma(\widehat{I_2(5)})$ induces a tessellation of the **hyperbolic plane** $\mathbb{H}^2$ (associated to the Tits form of $\widehat{I_2(5)}$) which projects on the Poincaré disk as follows:

(a) $\mathbb{H}^2$ and the Poincaré disk.



(b) The tessellation $\Sigma(\widehat{I_2(5)})$.

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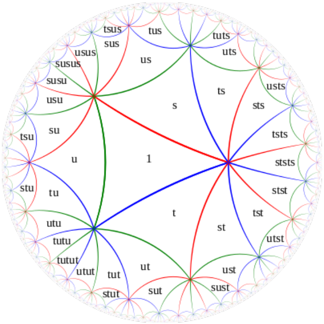
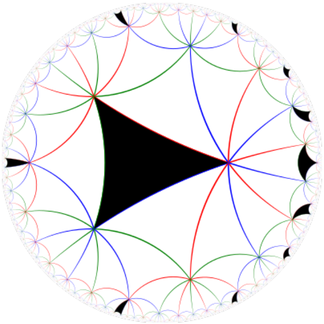


(c) Alcoves and walls for $\widehat{I_2(5)}$.



(d) Q-orbit of the central triangle.

The surface $\mathbf{T}(I_2(5))$ is obtained by gluing the triangles of a same orbit e.g. the green ones in the last figure.

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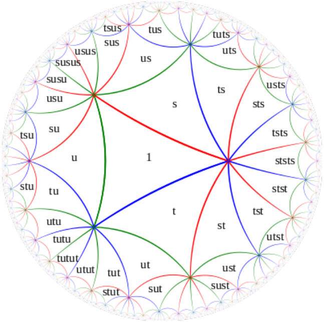

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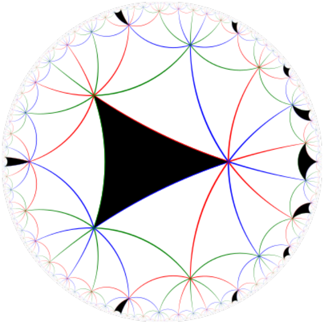

(d) Q-orbit of the central triangle.

Let $I_2(5) = \left\langle s, t \;\middle|\; s^2 = t^2 = (st)^5 = 1 \right\rangle$. The complex $C_*^{\mathrm{cell}}(\mathbf{T}(I_2(5)), I_2(5); \mathbb{Z})$ is

$$\mathbb{Z}[I_2(5)] \longrightarrow \mathbb{Z}[I_2(5)/\langle t \rangle] \oplus \mathbb{Z}[I_2(5)/\langle s^{ts} \rangle] \oplus \mathbb{Z}[I_2(5)/\langle s \rangle] \longrightarrow \mathbb{Z}^3.$$

Properties of $\mathbf{T}(W)$

Theorem (Problem B, G. 2023)

The space $\mathbf{T}(W)$ is a W-triangulated orientable compact Riemannian manifold and $\mathbf{T}(W) \simeq K(Q,1) \simeq B_Q$. If W is a Weyl group, then $\mathbf{T}(W)$ is a torus and otherwise, $\mathbf{T}(W)$ is hyperbolic.

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Remark

The manifold $\mathbf{T}(H_4)$ is the Davis hyperbolic 4-manifold (1985) and $\mathbf{T}(H_3)$ is the Zimmermann hyperbolic 3-manifold (1993). Their Betti numbers are $b_*(\mathbf{T}(H_3)) = (1, 11, 11, 1)$ and $b_*(\mathbf{T}(H_4)) = (1, 24, 72, 24, 1)$.

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We give a presentation of $\pi_1(\mathbf{T}(W)) \simeq Q$ and describe the W-dg-ring of $\mathbf{T}(W)$, which is the one we want.

Properties of $\mathbf{T}(W)$

Let $\mathbb{Q}_W$ be a splitting field for W. We can take

$$\mathbb{Q}_{I_2(m)} = \mathbb{Q}(\cos(2\pi/m)) \quad \text{and} \quad \mathbb{Q}_{H_3} = \mathbb{Q}_{H_4} = \mathbb{Q}(\sqrt{5}).$$

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If $W(q)$ (resp. $\widehat{W}(q)$) is the Poincaré series of W (resp. of $\widehat{W}$) then, as for tori,

$$\chi(\mathbf{T}(W)) = \left.\frac{W(q)}{\widehat{W}(q)}\right|_{q=1}.$$

Further details on the hyperbolic surfaces $\mathbf{T}(I_2(m))$

Corollary

Let $g \in \mathbb{N}^*$. Then $\mathbf{T}(I_2(2g+1))$, $\mathbf{T}(I_2(4g))$ and $\mathbf{T}(I_2(4g+2))$ are arithmetic Riemann surfaces with the same genus g.

We have an isomorphism

$$\mathbf{T}(I_2(4g+2)) \simeq \mathbf{T}(I_2(2g+1)),$$

and these two are not isomorphic to $\mathbf{T}(I_2(4g))$.

In particular, for $g = 1$, these are rational elliptic curves: the orbifold points in the Dirichlet domain of $PSL_2(\mathbb{Z})$.

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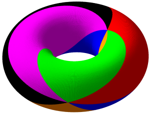
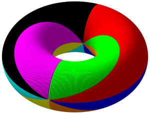
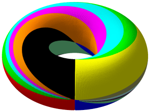
$\rightsquigarrow$ unusual point of view on tori!

Thank you very much!