Equivariant cellular structures on spheres and flag manifolds

Algebra/Topology Seminar, University of Copenhagen

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Menu

The problem

Notation

- G a connected reductive complex algebraic group,
- B a Borel subgroup of G.
- **Flag manifold** is the homogeneous space $\mathcal{F}\ell_{\mathbb{C}}(G) := G/B$.
- The **Weyl group** W acts freely on it.
- T maximal torus in G, then $G/T \circlearrowleft W$ and G/T is homotopy equivalent to G/B.
- In fact, if K maximal compact subgroup of G and $T_K := K \cap T \simeq U(1)^r$, then

$$\mathcal{F}\ell_{\mathbb{C}}(G) := G/B \overset{\text{diff}}{\simeq} K/T_K \circlearrowleft W = N_K(T_K)/T_K \simeq N_G(T)/T$$

- **Question**: Describe $R\Gamma(G/B, \mathbb{Z}) \in D^b(\mathbb{Z}W)$.
- **More precisely**: Describe $G/B = K/T_K$ as a W-equivariant cellular/simplicial complex.

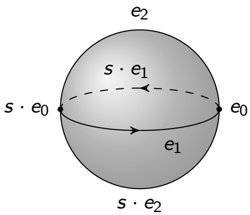
Example: type A

- $G = SL_n(\mathbb{C})$, $K = SU(n)$, $T = S(U(1)^n)$ and $W = \mathfrak{S}_n$,
- $B = \begin{pmatrix} * & * & \dots & * \\ 0 & * & & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & * \end{pmatrix}$ Borel subgroup,
- $G/B = \{\text{flags } (0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = \mathbb{C}^n)\}$,
- $K/T = \{\text{decompositions } \mathbb{C}^n = L_1 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} L_n\} \circlearrowleft \mathfrak{S}_n$,
- $G/T^{\mathbb{C}} = \{\text{decompositions } \mathbb{C}^n = L_1 \oplus \cdots \oplus L_n\} \circlearrowleft \mathfrak{S}_n$.

The example of SL_2

$$\mathcal{F}\ell(SL_2) \simeq SU(2)/T \simeq \mathbb{CP}^1 \circlearrowleft \mathfrak{S}_2 = \langle s \rangle\,,\ s(L_1 \overset{\perp}{\oplus} L_2) = (L_2 \overset{\perp}{\oplus} L_1).$$

$s \cdot [1 : z] = [-\overline{z} : 1] = [1 : -1/\overline{z}]$ antipode map!



$$\mathcal{C}_\bullet := \ \mathbb{Z}[\mathfrak{S}_2]\,\langle e_2\rangle \xrightarrow{1+s} \mathbb{Z}[\mathfrak{S}_2]\,\langle e_1\rangle \xrightarrow{1-s} \mathbb{Z}[\mathfrak{S}_2]\,\langle e_0\rangle$$

$$\mathrm{End}_{D^b(\mathbb{Z}\mathfrak{S}_2)}(\mathcal{C}_\bullet) = \mathbb{Z}[\mathfrak{S}_2]$$

The Borel picture for $H^*(G/B,\mathbb{Q})$

- Action of W on $H^*(G/B,\mathbb{Q})$ is well-known:

$$H^*(G/B,\mathbb{Q}) \simeq \mathbb{Q}[x_1,\dots,x_n]_W = \frac{\mathbb{Q}[x_1,\dots,x_n]}{\mathbb{Q}[x_1,\dots,x_n]_+^W},$$

Isomorphism induced by $x_i \mapsto c_1(\mathcal{L}_i)$, with $\mathcal{L}_i$ line bundle associated to the i^{th} fundamental weight.

The ungraded $\mathbb{Q}[W]$-module $H^*(G/B,\mathbb{Q})$ is the regular one.

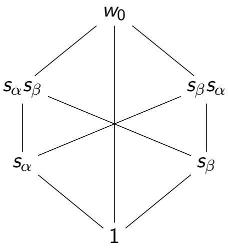
- The **Bruhat decomposition**

$$G/B = \bigsqcup_{w\in W} BwB/B$$

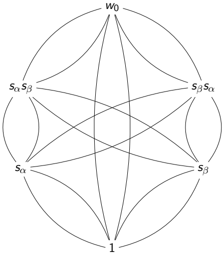
is cellular and
$\dim_{\mathbb{R}}(G/B) = \dim_{\mathbb{R}}(Bw_0B/B) = 2\ell(w_0) = 2|\Phi^+|$, with $w_0 \in W$ the longest element and Φ the root system.

The Goresky-Kottwitz-MacPherson graph

$\{\text{vertices}\} \leftrightarrow \{T\text{-fixed points}\}$, $\{\text{edges}\} \leftrightarrow \{T\text{-orbits of dim } 1\}$,
each edge $\leftrightarrow$ $\overline{T\text{-orbit}} = \mathbb{CP}^1$



(a) The GKM graph



(b) Many SL_2 situations

$\rightsquigarrow$ 1-skeleton and part of the 2-skeleton, like for SL_2.

Educated guess/hope

- Hope the complex has nice general combinatorial description.
- Hard to find a CW-structure. Try to guess ranks of free modules. Let

$$P_W^{\mathbb{C}}(q) := \sum_i \#\{W\text{-orbits of } k\text{-cells of } \mathcal{F}\ell_{\mathbb{C}}(G)\} q^i$$

  and similarly consider $P_W^{\mathbb{R}}$ for the real points. Constraints: $\deg(P_W^{\mathbb{C}}) = 2|\Phi^+|$ and $P_W^{\mathbb{C}}(-1) = 1$.
- Parametrization

$$\{k\text{-cells of } \mathcal{F}\ell_{\mathbb{R}}(G)\} \leftrightarrow \{k\text{-subsets of positive roots}\}.$$

  k-cells parametrized by k real parameters (one for each root).
  This would give $P_W^{\mathbb{R}}(q) = [2]_q^{|\Phi^+|}$.
  Recall that $[k]_q = 1 + q + \cdots + q^{k-1}$.

Educated guess/hope

- Missing cells in $\mathcal{F}\ell_{\mathbb{C}}(G)$: allow some parameters to take complex values.

  Compatible with GKM.

  Each positive root would have a multiplicity 0, 1 or 2

  $\rightsquigarrow$ multiset of positive roots with multiplicity.

  This gives $P_W^{\mathbb{C}}(q) = [3]_q^{|\Phi^+|}$.

  $\rightsquigarrow$ combinatorial flavour of the DeConcini-Salvetti complex.

  Recall that the DeConcini-Salvetti complex is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[W]$, with W finite Coxeter group, constructed using increasing chains of subsets of simple reflections.

  For SL_3, $P_W^{\mathbb{R}}(q) = [2]_q^3 = q^3 + 3q^2 + 3q + 1$ and $P_W^{\mathbb{C}}(q) = [3]_q^3 = q^6 3q^5 + 6q^4 + 7q^3 + 6q^2 + 3q + 1$.

Educated guess/hope

- Another possible formula (involving only simple roots) for this number of orbits: $\prod_i [2d_i - 1]_q$, with $(d_i)_i$ the degrees of W. Recall that the d_i's are the degrees of fundamental invariants of W and satisfy

$$\sum_i (d_i - 1) = |\Phi^+| \text{ and } \prod_i d_i = |W|.$$

  Hence $\deg\left(\prod_i [2d_i - 1]_q\right) = \sum_i (2d_i - 2) = 2|\Phi^+|$ and $\prod_i [2d_i - 1]_{-1} = 1$.
  Over $\mathbb{R}$, similar considerations would give $\prod_i [d_i]_q$.
  For SL_3, gives $P_W^{\mathbb{R}}(q) = [2]_q[3]_q = q^3 + 2q^2 + 2q + 1$ and $P_W^{\mathbb{C}}(q) = [3]_q[5]_q = q^6 + 2q^5 + 3q^4 + 3q^3 + 3q^2 + 2q + 1$.

- $[3]_q^{|\Phi^+|}$ has a clear link with GKM and easier to pass from $\mathbb{R}$ to $\mathbb{C}$, but $\prod_i [2d_i - 1]_q$ yields fewer cells.

Educated guess/hope

We summarize the guess for the polynomials $P_W^{\mathbb{R}}$ and $P_W^{\mathbb{C}}$ in the following table

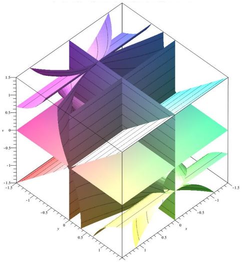
$P_W^{\mathbb{C}}(q)$	$[3]_q^{\lvert\Phi^+\rvert}$	$\prod_i [2d_i - 1]_q$
$P_W^{\mathbb{R}}(q)$	$[2]_q^{\lvert\Phi^+\rvert}$	$\prod_i [d_i]_q$

$\mathcal{F}\ell_{\mathbb{R}}(SL_3) \hookrightarrow \mathbb{P}(\mathcal{O}_{\min})(\mathbb{R}) \subset \mathbb{P}(\mathfrak{sl}_3(\mathbb{R})) = \mathbb{P}^7(\mathbb{R})$

Easier on $\mathbb{R}$ because W-action is algebraic and $\dim_{\mathbb{R}} = 3$ (not 6).

Starting with the GKM graph, we found an equivariant decomposition of $\mathcal{F}\ell_{\mathbb{R}}(SL_3)$, with ranks $4, 6, 3, 1$.

Doesn't satisfy the hope and complex homotopy equivalent to a smaller term with ranks $1, 3, 3, 1$ and even $1, 2, 2, 1$.



New look at SL_3/B

We look at $\mathcal{F}\ell_{\mathbb{R}}(SL_3) = SO(3)/S(O(1)^3) = SO(3)/\{\pm 1\}^2 \circlearrowleft \mathfrak{S}_3$.

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[d, "/\{\pm1\}"] \arrow[dd, bend right, "/\mathcal{Q}_8"'] & \circlearrowleft \mathcal{Q}_8 \rtimes \mathfrak{S}_3 = \mathcal{O} \\
SO(3) \arrow[d, "/\{\pm1\}^2"] & \circlearrowleft \{\pm1\}^2 \rtimes \mathfrak{S}_3 = W(D_3) = \mathfrak{S}_4 \\
SO(3)/\{\pm1\}^2 \arrow[d, "/\mathfrak{S}_3"] & \circlearrowleft \mathfrak{S}_3 \\
\mathbb{S}^3/\mathcal{O} & \circlearrowleft 1
\end{tikzcd}$$

This last space is called a **spherical space form**.

Finite groups acting freely on spheres

Theorem (*Hopf 1925, Milnor 1957*)

Any finite group acting freely and isometrically on $\mathbb{S}^3$ is isomorphic to one of the following groups:

1. 1, $\mathcal{O}$, $\mathcal{I}$, *quaternion groups* $\mathcal{Q}_{8n} = \langle x, y \mid x^2 = (xy)^2 = y^{2n} \rangle$,
2. *generalized dihedral groups*

$$\mathcal{D}_{2^k(2n+1)} = \left\langle x, y \mid x^{2^k} = y^{2n+1} = 1,\ xyx^{-1} = y^{-1} \right\rangle,\ k \geq 2,\ n \geq 1,$$

3. *generalized tetrahedral groups (including $\mathcal{T} = P'_{24}$)*

$$P'_{8\cdot 3^k} = \left\langle x, y, z \mid x^2 = (xy)^2 = y^2,\ zxz^{-1} = y,\ zyz^{-1} = xy,\ z^{3^k} = 1 \right\rangle,\ k \geq 1,$$

4. *The product of any of these groups with a cyclic group of relatively prime order.*

Orbit polytopes

Free isometric action of a finite group $G \circlearrowright \mathbb{S}^n \ni v_0$.

The **orbit polytope** of G is $P_G := \mathrm{conv}(G \cdot v_0)$.

G acts on P_G and on its faces.

This action is free and the projection

$$\partial P_G \to \mathbb{S}^n$$

is a G-homeomorphism.

Theorem (*Fêmina–Galves–Manzoli Neto–Spreafico (2013), Chirivì–Spreafico (2017)*)

Assume that $\mathrm{span}(G \cdot v_0) = \mathbb{R}^{n+1}$. *Then there is a system* $F_1, \ldots, F_r$ *of orbit representatives for the* G*-action on facets of* P_G *such that* $\bigcup_i F_i$ *is a fundamental domain for* G *on* ∂P_G.

Binary polyhedral groups

Besides cyclic and (binary) dihedral groups, $\mathbb{S}^3$ has three finite subgroups: the **binary poyhedral groups**. These are

$$\mathcal{T} := \left\langle i, \frac{-1+i+j+k}{2} \right\rangle, \quad \mathcal{O} := \left\langle \mathcal{T}, \frac{1+i}{\sqrt{2}} \right\rangle, \quad \mathcal{I} := \left\langle \mathcal{T}, \frac{\phi^{-1}+i+\phi j}{2} \right\rangle,$$

where $\phi = \frac{1+\sqrt{5}}{2}$.

The polytopes $P_{\mathcal{T}}$, $P_{\mathcal{O}}$ and $P_{\mathcal{I}}$ are respectively known as the *24-cell*, the *disphenoidal 288-cell* and the *600-cell*.

Chirivì-Spreafico's method to $P_{\mathcal{O}}$ yields a polyhedral fundamental domain for $\mathcal{O}$ on $\partial P_{\mathcal{O}} = \mathbb{S}^3$, which we have to decompose.

The chain complexes

Theorem (*Chirivì–G.–Spreafico, 2020*)

The sphere $\mathbb{S}^3$ admits an $\mathcal{O}$-equivariant cellular decomposition whose associated cellular homology complex is

$$\mathbb{Z}[\mathcal{O}] \xrightarrow[\partial_3]{\begin{pmatrix} 1-\tau_i & 1-\tau_j & 1-\tau_k \end{pmatrix}} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow[\partial_2]{\begin{pmatrix} \omega_i & 1 & \tau_j - 1 \\ \tau_k - 1 & \omega_j & 1 \\ 1 & \tau_i - 1 & \omega_k \end{pmatrix}} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow[\partial_1]{\begin{pmatrix} \tau_i - 1 \\ \tau_j - 1 \\ \tau_k - 1 \end{pmatrix}} \mathbb{Z}[\mathcal{O}] \ ,$$

where

$$\omega_0 = \frac{1-i-j-k}{2}, \ \omega_i = \frac{1+i-j-k}{2}, \ \omega_j = \frac{1-i+j-k}{2}, \ \omega_k = \frac{1-i-j+k}{2},$$

$$\tau_i = \frac{1-i}{\sqrt{2}}, \ \tau_j = \frac{1-j}{\sqrt{2}}, \ \tau_k = \frac{1-k}{\sqrt{2}}.$$

The chain complexes

Theorem (*Chirivì–G.–Spreafico, 2020*)

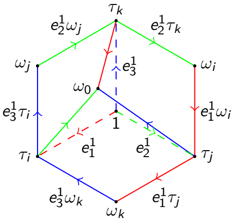
The sphere $\mathbb{S}^3$ admits an $\mathcal{I}$-equivariant cellular decomposition whose associated cellular homology complex is

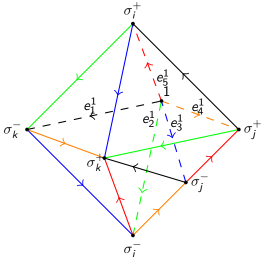
$$\mathbb{Z}[\mathcal{I}] \xrightarrow[\partial_3]{{}^t\begin{pmatrix} \sigma_i^- - 1 \\ \sigma_j^- - 1 \\ \sigma_j^+ - 1 \\ \sigma_i^+ - 1 \\ \sigma_k - 1 \end{pmatrix}} \mathbb{Z}[\mathcal{I}]^5 \xrightarrow[\partial_2]{\begin{pmatrix} \sigma_j^+ & 0 & 0 & 1 & -1 \\ -1 & \sigma_i^+ & 0 & 0 & 1 \\ 1 & -1 & \sigma_k & 0 & 0 \\ 0 & 1 & -1 & \sigma_i^- & 0 \\ 0 & 0 & 1 & -1 & \sigma_j^- \end{pmatrix}} \mathbb{Z}[\mathcal{I}]^5 \xrightarrow[\partial_1]{\begin{pmatrix} \sigma_k - 1 \\ \sigma_i^- - 1 \\ \sigma_j^- - 1 \\ \sigma_j^+ - 1 \\ \sigma_i^+ - 1 \end{pmatrix}} \mathbb{Z}[\mathcal{I}] \ ,$$

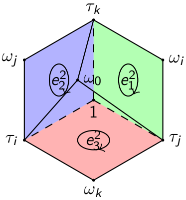
where

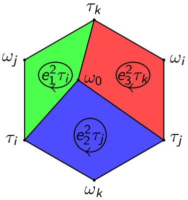
$$\sigma_i^{\pm} = \frac{\phi - \phi^{-1} i \pm j}{2}, \ \sigma_j^{\pm} = \frac{\phi \pm \phi^{-1} j + k}{2}, \ \sigma_k = \frac{\phi - i - \phi^{-1} k}{2}.$$

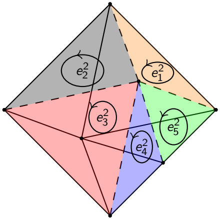
What does it look like?

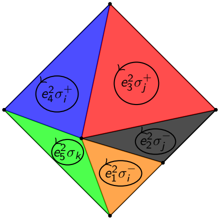












Application to $H^*(\mathcal{O}, \mathbb{Z})$ and $H^*(\mathcal{I}, \mathbb{Z})$

Corollary

With the notations of the previous result, for $q \geq 1$ and $G \in \{\mathcal{O}, \mathcal{I}\}$, we let

$$\partial_{4q-3} := \partial_1, \quad \partial_{4d-2} := \partial_2, \quad \partial_{4q-1} := \partial_3, \quad \partial_{4q} := \left(\sum_{g \in G} g \right).$$

The following complex is a 4-periodic resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$

$$\cdots \xrightarrow{\partial_{4q-2}} \mathbb{Z}[G]^k \xrightarrow{\partial_{4q-3}} \mathbb{Z}[G] \xrightarrow{\partial_{4q-4}} \cdots \xrightarrow{\partial_2} \mathbb{Z}[G]^k \xrightarrow{\partial_1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z}$$

where $k = 3$ for $G = \mathcal{O}$ and $k = 5$ for $G = \mathcal{I}$.

Taking the direct limit $\mathbb{S}^\infty = \varinjlim \mathbb{S}^{4n-1}$, we obtain an equivariant cell decomposition of the universal G-bundle, built inductively using "curved joins", starting with $\mathbb{S}^3$.

Application to $H^*(\mathcal{O}, \mathbb{Z})$ and $H^*(\mathcal{I}, \mathbb{Z})$

We recover that $\mathbb{S}^3/\mathcal{I}$ is a homology sphere (the Poincaré sphere).

The 4-periodic resolutions found for $\mathcal{O}$ and $\mathcal{I}$ allow to compute their cohomology.

Corollary (*Tomoda–Zvengrowski 2008, Chirivì–G.–Spreafico 2020*)

The integral group cohomology of $\mathcal{O}$ (resp. $\mathcal{I}$) is as follows:

$$\begin{cases} H^q(\mathcal{O}, \mathbb{Z}) = \mathbb{Z} & \textit{if } q = 0, \\ H^q(\mathcal{O}, \mathbb{Z}) = \mathbb{Z}/48\mathbb{Z} & \textit{if } 0 < q \equiv 0[4], \\ H^q(\mathcal{O}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} & \textit{if } q \equiv 2[4], \\ H^q(\mathcal{O}, \mathbb{Z}) = 0 & \textit{otherwise} \end{cases} \quad \textit{resp.} \quad \begin{cases} H^q(\mathcal{I}, \mathbb{Z}) = \mathbb{Z} & \textit{if } q = 0, \\ H^q(\mathcal{I}, \mathbb{Z}) = \mathbb{Z}/120\mathbb{Z} & \textit{if } 0 < q \equiv 0[4] \\ H^q(\mathcal{I}, \mathbb{Z}) = 0 & \textit{otherwise} \end{cases}$$

The cellular complex of $\mathbb{Z}[\mathfrak{S}_3]$-modules of $\mathcal{F}\ell_{\mathbb{R}}(SL_3)$

Theorem (*Chirivì–G.–Spreafico, 2020*)

The real flag manifold $\mathcal{F}\ell_{\mathbb{R}}(SL_3)$ admits an $\mathfrak{S}_3$-equivariant cell structure with cellular chain complex $\mathcal{C}_\bullet^{\mathbb{R}}$ given by

$$\mathbb{Z}[\mathfrak{S}_3] \xrightarrow[\partial_3]{\begin{pmatrix} 1-s_\beta & 1-w_0 & 1-s_\alpha \end{pmatrix}} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow[\partial_2]{\begin{pmatrix} s_\alpha s_\beta & 1 & w_0-1 \\ s_\alpha-1 & s_\alpha s_\beta & 1 \\ 1 & s_\beta-1 & s_\alpha s_\beta \end{pmatrix}} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow[\partial_1]{\begin{pmatrix} 1-s_\beta \\ 1-w_0 \\ 1-s_\alpha \end{pmatrix}} \mathbb{Z}[\mathfrak{S}_3] \ ,$$

where $w_0 = s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$ is the longest element of $\mathfrak{S}_3$.

Homotopic to a complex with ranks $1, 2, 2, 1$, but we have found no geometric model for it.

Using GAP4 and CAP, we compute

$$\mathrm{End}_{D^b(\mathbb{Z}\mathfrak{S}_3)}(\mathcal{C}_\bullet^{\mathbb{R}}) = \mathrm{End}_{K^b(\mathbb{Z}\mathfrak{S}_3)}(\mathcal{C}_\bullet^{\mathbb{R}}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}[X]/(X^2-9).$$

The $\mathbb{Z}[\mathfrak{S}_3]$-module structure on cohomology

We may dualize the above complex to obtain the action of $\mathfrak{S}_3$ on the cohomology of $\mathcal{F}\ell_{\mathbb{R}}(SL_3)$.

More precisely, we have the following result:

Corollary

The $\mathbb{Z}[\mathfrak{S}_3]$-module $H^i(\mathcal{F}\ell_{\mathbb{R}}(SL_3), \mathbb{Z})$ is either

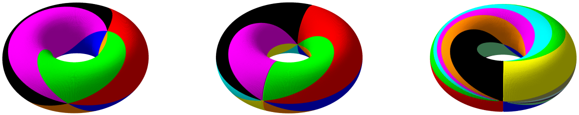
$$\begin{cases} \mathbb{Z} & \text{if } i = 0, 3 \\ \mathbf{2}_{\mathbb{F}_2} & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbb{Z}$ is the trivial module and $\mathbf{2}_{\mathbb{F}_2}$ is the irreducible $\mathbb{F}_2[\mathfrak{S}_3]$-module of degree 2.

cf Rabelo–San Martin for $H^*(\mathcal{F}\ell_{\mathbb{R}}(G), \mathbb{Z})$ as a graded $\mathbb{Z}$-module.

- Cells of $\mathcal{F}\ell_{\mathbb{R}}(SL_3)$ constructed from geodesics of $\mathbb{S}^3$ and in fact (open) geodesic simplices in $\mathcal{F}\ell_{\mathbb{R}}(SL_3)$.
  Plan for $SL_n(\mathbb{R})$: define geodesic simplices on $SO(n)$.
- Many constraints on the complex: ranks, torsion-free homology, characters, Euler characteristic, maybe Poincaré duality... We could directly guess the complex, using GAP and CAP for instance. Package developed with S. Posur to deal with free $\mathbb{Z}[G]$-modules.
- G/B is the 0-fiber of the Springer sheaf $\mathcal{K}$. It is used in Borho-MacPherson's proof of $\mathrm{End}(\mathcal{K}_0) = \mathbb{Q}[W]$.
  W acts on the cohomology of other Springer fibers. Once the problem is solved for G/B, we could try with other fibers e.g. by finding homotopy equivalent spaces on which W acts.
- Work in progress: the $\mathbb{Z}[W]$-complex of the torus and generalization to compact hyperbolic Coxeter groups.
- Classifying space B_T of the torus T: $W \circlearrowright T$ implies $W \circlearrowright B_T \simeq (\mathbb{CP}^\infty)^r$. Equivariant cell structure on B_T?

Thank you !



The complex for the first decomposition is

$$\mathbb{Z}[\mathfrak{S}_3]^4 \xrightarrow{d_3} \mathbb{Z}[\mathfrak{S}_3]^6 \xrightarrow{d_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{d_1} \mathbb{Z}[\mathfrak{S}_3] \ ,$$

where

$$d_1 = \begin{pmatrix} 1 - s_\alpha & 1 - s_\beta & 1 - w_0 \end{pmatrix}, \ d_3 = \begin{pmatrix} 0 & s_\alpha & 0 & 1 \\ -s_\beta s_\alpha & 0 & -w_0 & 0 \\ 0 & s_\beta s_\alpha & 1 & 0 \\ 1 & 0 & 0 & s_\beta s_\alpha \\ -s_\alpha s_\beta & s_\alpha s_\beta & 0 & 0 \\ 0 & 0 & s_\alpha s_\beta & -s_\alpha s_\beta \end{pmatrix},$$

$$d_2 = \begin{pmatrix} -1 & 1 & 1 & s_\alpha & w_0 - s_\alpha s_\beta & s_\beta - s_\beta s_\alpha \\ s_\beta s_\alpha - s_\beta & s_\alpha - 1 & -w_0 & w_0 & s_\alpha s_\beta & s_\alpha s_\beta \\ s_\beta & s_\beta s_\alpha & s_\alpha - 1 & s_\alpha s_\beta - w_0 & -s_\beta & s_\beta s_\alpha \end{pmatrix}.$$