# Equivariant cellular models in Lie theory

## Ph.D. thesis

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## $R\Gamma(X,\mathbb{Z}) \in \mathcal{D}^b(\mathbb{Z}[W])$ and equivariant cellular structures

**discrete** group $W \circlearrowright X$ topological space

$$\rightsquigarrow \quad W \circlearrowright H^*(X,\mathbb{Z}) = H^*(R\Gamma(X,\mathbb{Z})).$$

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## Definition

A CW-structure on $X$ is $W$-**equivariant** if

- $W$ acts on cells
- For $e \subset X$ a cell and $w \in W$, if $we = e$ then $w_{|e} = \mathrm{id}_e$.

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## Theorem

The complex $C^*_{\mathrm{cell}}(X, W; \mathbb{Z})$ is well-defined up to homotopy and $C^*_{\mathrm{cell}}(X, W; \mathbb{Z}) \cong R\Gamma(X,\mathbb{Z})$ in $\mathcal{D}^b(\mathbb{Z}[W])$.

# Illustration: $\{\pm 1\} \circlearrowright \mathbb{S}^2 \subset \mathbb{R}^3$

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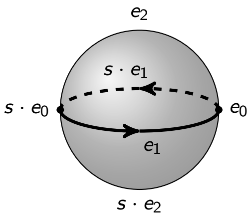


Chain complex given by

$$C_*^{\mathrm{cell}}(\mathbb{S}^2, C_2; \mathbb{Z}) = \left( \mathbb{Z}[C_2] \langle e_2 \rangle \xrightarrow{1+s} \mathbb{Z}[C_2] \langle e_1 \rangle \xrightarrow{1-s} \mathbb{Z}[C_2] \langle e_0 \rangle \right)$$

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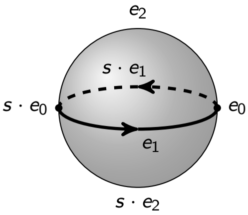


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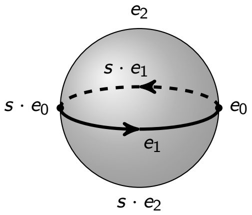


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so $R\Gamma(\mathbb{S}^2, \underline{\mathbb{Q}}) \simeq \mathbb{1} \oplus \varepsilon[-2]$ and $R\Gamma(\mathbb{S}^2, \underline{\mathbb{F}_2})$ has cohomology $H^*(\mathbb{S}^2, \mathbb{F}_2) = \mathbb{1} \oplus \mathbb{1}[-2]$, however, $R\Gamma(\mathbb{S}^2, \underline{\mathbb{F}_2})$ is indecomposable...

# Tori: the simply-connected case

$K$ simple compact Lie group of rank $n$ such that $\pi_1(K) = 1$,
$T \simeq (\mathbb{S}^1)^n$ maximal torus, $W := N_K(T)/T$ Weyl group.
$W \circlearrowright T$, e.g. $\mathfrak{S}_{n+1} \circlearrowright S(U(1)^{n+1}) < SU(n+1)$.

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*Exhibit a $W$-equivariant triangulation of $T$.*

$\mathfrak{t}/\Lambda \xrightarrow[\exp]{\sim} T$ where $\mathfrak{t} := \mathrm{Lie}(T)$ and $\Lambda =$ **cocharater lattice**.
$\rightsquigarrow$ $W_\Lambda$-triangulation for $W_\Lambda := \Lambda \rtimes W \circlearrowright \mathfrak{t}$.
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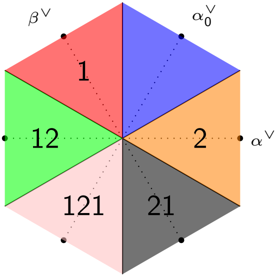
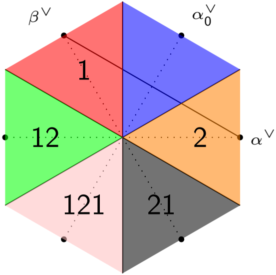
**Theorem (A1)**

*The fundamental alcove induces a $W_{\mathrm{a}}$-triangulation of $\mathfrak{t}$, yielding a $W$-triangulation of $T$. The $W$-dg-ring $C_*^{\mathrm{cell}}(T, W; \mathbb{Z})$ is described in terms of **parabolic cosets** of $W_{\mathrm{a}}$ and **deflation** $\mathrm{Def}_W^{W_{\mathrm{a}}}$.*

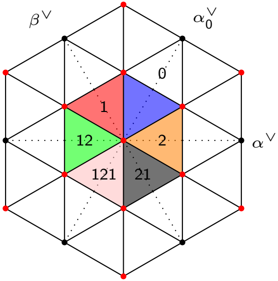
# Example in type $A_2$




(a) Fundamental chamber (in blue) and its $\mathfrak{S}_3$-translates.




(b) What if we add a wall?




(c) Alcoves for $(\mathfrak{S}_3)_{\mathrm{a}} = \langle 1, 2, 0 \rangle$.

Figure: Chambers subdivided into alcoves.

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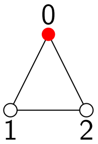
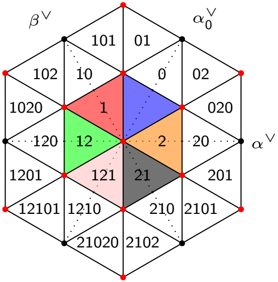
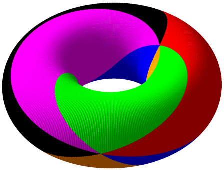



(a) Dynkin diagram of $\widetilde{A}_2$.




(b) Fundamental alcove and some of its $(\mathfrak{S}_3)_{\mathrm{a}}$-translates.

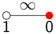
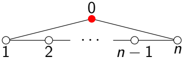
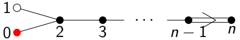
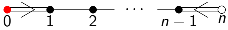
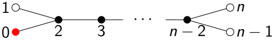
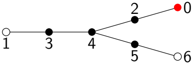
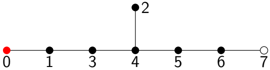
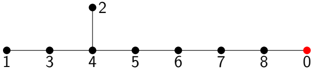
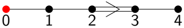
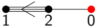


(c) Resulting $\mathfrak{S}_3$-triangulation of $S(U(1)^3) \simeq (\mathbb{S}^1)^2$.

Figure: Triangulation of the torus $S(U(1)^3)$ of $SU(3)$.

The complex $C_*^{\mathrm{cell}}(S(U(1)^3), \mathfrak{S}_3; \mathbb{Z})$ is given by

$$\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}} \mathbb{Z}[\mathfrak{S}_3/\langle s_\beta \rangle] \oplus \mathbb{Z}[\mathfrak{S}_3/\langle s_\alpha s_\beta s_\alpha \rangle] \oplus \mathbb{Z}[\mathfrak{S}_3/\langle s_\alpha \rangle] \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}} \mathbb{Z}^3.$$

| Type | Affine Dynkin diagram | Fundamental group $\Omega \simeq P/Q$ |
|---|---|---|
| $\widetilde{A}_1$ |  | $\mathbb{Z}/2\mathbb{Z}$ |
| $\widetilde{A}_n$ $(n \geq 2)$ |  | $\mathbb{Z}/(n+1)\mathbb{Z}$ |
| $\widetilde{B}_n$ $(n \geq 3)$ |  | $\mathbb{Z}/2\mathbb{Z}$ |
| $\widetilde{C}_n$ $(n \geq 2)$ |  | $\mathbb{Z}/2\mathbb{Z}$ |
| $\widetilde{D}_n$ $(n \geq 4)$ |  | $\begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ even} \\ \mathbb{Z}/4\mathbb{Z} & \text{if } n \text{ is odd} \end{cases}$ |
| $\widetilde{E}_6$ |  | $\mathbb{Z}/3\mathbb{Z}$ |
| $\widetilde{E}_7$ |  | $\mathbb{Z}/2\mathbb{Z}$ |
| $\widetilde{E}_8$ |  | 1 |
| $\widetilde{F}_4$ |  | 1 |
| $\widetilde{G}_2$ |  | 1 |

# The general case: barycentric subdivision

Other extreme case: the adjoint group, i.e. $Y = P^\vee$ is the *coweight lattice* and $\widehat{W}_{\mathrm{a}} = P^\vee \rtimes W$ is the **extended affine Weyl group**.

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## Theorem (A2)

*The barycentric subdivision of the fundamental alcove induces a $\widehat{W_{\mathrm{a}}}$-equivariant triangulation of $\mathfrak{t}$. The same holds for any $W$-lattice $Q^\vee \subset \Lambda \subset P^\vee$ and the intermediate group $W_\Lambda$.*

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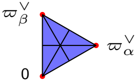


$\mathrm{Stab}_{\widehat{W_\mathrm{a}}}(\mathcal{A}) = \{1, \omega_\alpha, \omega_\beta\} \simeq \mathbb{Z}/3\mathbb{Z}$ with $\omega_\beta$ rotation of the triangle with angle $2\pi/3$. The complex for $P(S(U(1)^3)) < PSU(3)$ is

$$\mathbb{Z}[\mathfrak{S}_3]^2 \xrightarrow{\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & s_\beta s_\alpha \end{pmatrix}} \mathbb{Z}[\mathfrak{S}_3/\langle s_\beta \rangle] \oplus \mathbb{Z}[\mathfrak{S}_3/\langle s_\alpha \rangle] \oplus \mathbb{Z}[\mathfrak{S}_3]^2 \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ -1 & s_\beta s_\alpha & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}[\mathfrak{S}_3/\langle s_\beta \rangle] \oplus \mathbb{Z}[\mathfrak{S}_3/\langle s_\alpha s_\beta \rangle].$$

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"True tori": $W$ Weyl, $r_W = s_{\widetilde{\alpha}}$ (highest root), $\widehat{W} = W_{\mathrm{a}}$.

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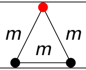
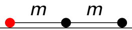
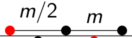
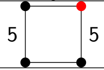
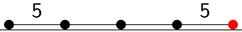
| Extension | Coxeter graph |
|---|---|
| $\widehat{I_2(m)}$ $(m \equiv 1[2])$ |  |
| $\widehat{I_2(m)}$ $(m \equiv 0[4])$ |  |
| $\widehat{I_2(m)}$ $(m \equiv 2[4])$ |  |
| $\widehat{H_3}$ |  |
| $\widehat{H_4}$ |  |

## The non-commutative lattice $Q$ and the manifold $\mathbf{T}(W)$

Sending $\widehat{s}_0 \in \widehat{W}$ to $r_W \in W$ induces a surjection $\pi : \widehat{W} \twoheadrightarrow W$ and we have $\widehat{W} = Q \rtimes W$, where the torsion-free subgroup

$$Q := \ker(\pi) = \left\langle (\widehat{s}_0 r_W)^{\widehat{W}} \right\rangle \lhd \widehat{W}$$

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is $\mathbb{Z}\Phi^\vee$ in the crystallographic case and a non-commutative analogue otherwise. Consider the **Coxeter complex**

$$\Sigma(\widehat{W}) := \left( \bigcup_{w \in \widehat{W}} w(\overline{C} \setminus \{0\}) \right) / \mathbb{R}_+^*,$$

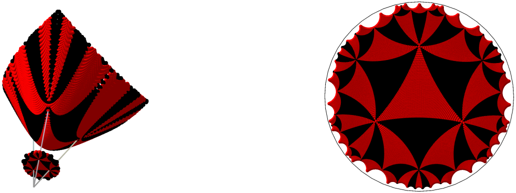
where $C$ is the *fundamental chamber* of $\widehat{W}$. We define

$$\mathbf{T}(W) := \Sigma(\widehat{W})/Q.$$

This is a maximal torus in the crystallographic case and an "analogue" otherwise.

# The example of $I_2(5)$

For $W = I_2(5)$, the simplicial structure of $\Sigma(\widehat{I_2(5)})$ induces a tessellation of the **hyperbolic plane** $\mathbb{H}^2$ (associated to the Tits form of $\widehat{I_2(5)})$ which projects on the Poincaré disk as follows:



(a) The plane $\mathbb{H}^2$ and the Poincaré disk.

(b) The tessellation $\Sigma(\widehat{I_2(5)})$.

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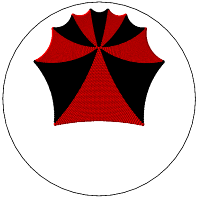
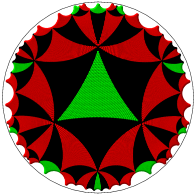


(c) Fundamental domain for $Q$.



(d) $Q$-orbit of the fundamental triangle.

The surface $\mathbf{T}(I_2(5))$ is obtained by gluing the triangles of a same orbit e.g. the green ones in the last figure.

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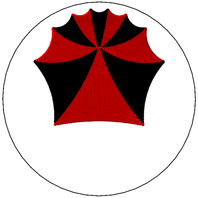

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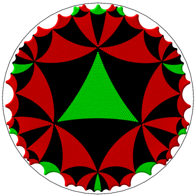

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Let $I_2(5) = \left\langle s_1, s_2 \;\middle|\; s_1^2 = s_2^2 = (s_1 s_2)^5 = 1 \right\rangle$. The complex $C_*^{\mathrm{cell}}(\mathbf{T}(I_2(5)), I_2(5); \mathbb{Z})$ is

$$\mathbb{Z}[I_2(5)] \xrightarrow{(1\;1\;-1)} \mathbb{Z}[I_2(5)/\langle\, s_2 \,\rangle] \oplus \mathbb{Z}[I_2(5)/\langle\, s_1^{s_2 s_1} \,\rangle] \oplus \mathbb{Z}[I_2(5)/\langle\, s_1 \,\rangle] \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}} \mathbb{Z}^3.$$

# Properties of $\mathbf{T}(W)$

**Theorem (A3)**

*The space $\mathbf{T}(W)$ is a $W$-triangulated orientable compact Riemannian manifold and $\mathbf{T}(W) \simeq K(Q,1) \simeq B_Q$. If $W$ is a Weyl group, then $\mathbf{T}(W)$ is a torus and otherwise, $\mathbf{T}(W)$ is hyperbolic.*

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## Example

$\mathbf{T}(I_2(2g+1))$, $\mathbf{T}(I_2(4g))$ and $\mathbf{T}(I_2(4g+2))$ are arithmetic Riemann surfaces of genus $g$ and rational elliptic curves for $g=1$.
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We give a presentation of $\pi_1(\mathbf{T}(W)) \simeq Q$ and describe the $W$-dg-ring of $\mathbf{T}(W)$, which is the one we want.

# Properties of $\mathbf{T}(W)$

## Proposition

*$H_* := H_*(\mathbf{T}(W), \mathbb{Z})$ is torsion-free, with palindromic Betti numbers (by Poincaré duality). We have $H_0 = \mathbb{1}$, $H_n = \operatorname{sgn}$ and the geometric representation of $W$ is a direct summand of $H_1 \otimes \mathbb{Q}_W$.*

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## Remark

*$\mathbf{T}(H_4)$ is the Davis hyperbolic 4-manifold (1985) and $\mathbf{T}(H_3)$ is the Zimmermann hyperbolic 3-manifold (1993). Their Betti numbers are $b_*(\mathbf{T}(H_3)) = (1, 11, 11, 1)$ and $b_*(\mathbf{T}(H_4)) = (1, 24, 72, 24, 1)$.*

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Finally, if $W(q)$ (resp. $\widehat{W}(q)$) is the Poincaré series of $W$ (resp. of $\widehat{W}$) then, as for tori,

$$\chi(\mathbf{T}(W)) = \left.\frac{W(q)}{\widehat{W}(q)}\right|_{q=1}.$$

# Second problem: flag manifolds

## Notation

- $G$ a connected reductive complex algebraic group,
- $B$ a Borel subgroup of $G$, $T^{\mathbb{C}} < G$ maximal torus such that $T^{\mathbb{C}} < B$.
- $W := N_G(T^{\mathbb{C}})/T^{\mathbb{C}}$ the **Weyl group**.
- **Flag manifold**: the homogeneous space $\mathcal{F}_G(\mathbb{C}) := G/B$.
- If $K$ maximal compact subgroup of $G$ and $T := K \cap T^{\mathbb{C}} \simeq (\mathbb{S}^1)^r$, then

$$\mathcal{F}_G(\mathbb{C}) := G/B \overset{\text{diff}}{\simeq} K/T \circlearrowleft W = N_K(T)/T.$$

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**Problem (B)**

*Describe $G/B = K/T$ as a $W$-equivariant CW-complex.*

# Example: type $A$

- $G = SL_n(\mathbb{C})$, $K = SU(n)$, $T = S(U(1)^n)$ and $W = \mathfrak{S}_n$,
- $B = \begin{pmatrix} * & * & \dots & * \\ 0 & * & & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & * \end{pmatrix}$ Borel subgroup,
- $G/B = \{\text{flags } (0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = \mathbb{C}^n)\}$,
- $K/T = \{\text{decompositions } \mathbb{C}^n = L_1 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} L_n\} \circlearrowleft \mathfrak{S}_n$,
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**Example**

$$\mathcal{F}_{SL_2}(\mathbb{C}) \simeq U(2)/U(1)^2 \simeq \mathbb{CP}^1 \circlearrowleft \mathfrak{S}_2 = \langle s \rangle,\ s(L_1 \overset{\perp}{\oplus} L_2) = L_2 \overset{\perp}{\oplus} L_1.$$

$$[1 : z] \cdot s = [-\bar{z} : 1] = [1 : -1/\bar{z}]$$

$\rightsquigarrow$ antipode on $\mathbb{S}^2$, as in the example of the introduction.

## A first decomposition of $\mathcal{F}_3(\mathbb{R}) := \mathcal{F}_{SL_3}(\mathbb{R}) = SL_3(\mathbb{R})/B$

$\mathcal{O}_{\min} := SL_3(\mathbb{C}) \cdot \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ **minimal nilpotent orbit**, then
$\mathcal{F}_3(\mathbb{C}) = \mathbb{P}(\overline{\mathcal{O}_{\min}}) \subset \mathbb{P}(\mathfrak{sl}_3) \simeq \mathbb{CP}^7$, so $\mathcal{F}_3(\mathbb{R}) \hookrightarrow \mathbb{RP}^7$.

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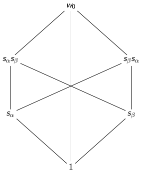
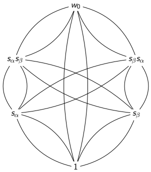
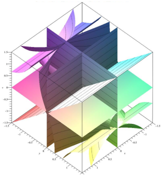



(a) GKM graph




(b) $SL_2$ situations



(c) 3-cells of $\mathcal{F}_3(\mathbb{R})$

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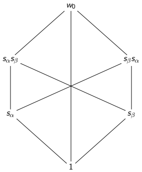


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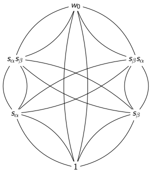


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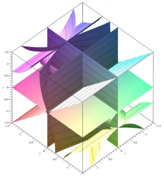

(c) 3-cells of $\mathcal{F}_3(\mathbb{R})$

**Theorem (B1)**

- *$\mathcal{F}_3(\mathbb{R})$ admits an $\mathfrak{S}_3$-cellular structure whose cellular chain complex has the shape $\mathbb{Z}[\mathfrak{S}_3]^4 \to \mathbb{Z}[\mathfrak{S}_3]^6 \to \mathbb{Z}[\mathfrak{S}_3]^3 \to \mathbb{Z}[\mathfrak{S}_3]$.*
- *There is an $\mathfrak{S}_3$-isomorphism $\mathbb{F}_2[x, y, z]_{\mathfrak{S}_3} \to H^*(\mathcal{F}_3(\mathbb{R}), \mathbb{F}_2)$ sending $x$, $y$ and $z$ to irreducible real algebraic 1-cocycles.*

# New look at $\mathcal{F}_3(\mathbb{R})$

Recall $\mathcal{F}_3(\mathbb{R}) = SO(3)/S(O(1)^3) = SO(3)/\{\pm 1\}^2 \circlearrowleft \mathfrak{S}_3$.

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\mathbb{S}^3/\mathcal{O} & \circlearrowleft 1
\end{tikzcd}$$

where $\mathcal{O}$ is the **binary octahedral group**. This last space is called a **spherical space form**. Construct an $\mathcal{O}$-cellular structure on $\mathbb{S}^3$?

# The complex $1, 3, 3, 1$ for $\mathcal{F}_3(\mathbb{R})$

## Theorem (B2)

*The real flag manifold $\mathcal{F}_3(\mathbb{R}) \simeq \mathbb{S}^3/\mathcal{Q}_8$ admits an $\mathfrak{S}_3$-equivariant cell structure with cellular chain complex given by*

$$\mathbb{Z}[\mathfrak{S}_3] \xrightarrow[\partial_3]{\begin{pmatrix} 1 - s_\beta & 1 - w_0 & 1 - s_\alpha \end{pmatrix}} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow[\partial_2]{\begin{pmatrix} s_\alpha s_\beta & 1 & w_0 - 1 \\ s_\alpha - 1 & s_\alpha s_\beta & 1 \\ 1 & s_\beta - 1 & s_\alpha s_\beta \end{pmatrix}} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow[\partial_1]{\begin{pmatrix} 1 - s_\beta \\ 1 - w_0 \\ 1 - s_\alpha \end{pmatrix}} \mathbb{Z}[\mathfrak{S}_3],$$

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## Remark

- *We also treat the case of the binary icosahedral group $\mathcal{I}$.*
- $\mathbb{S}^3 = \mathrm{Spin}(3)$ *but other* $\mathrm{Spin}(n)$ *groups are complicated.*
- *but the 1-cells of $\mathbb{S}^3$ and $\mathcal{F}_3(\mathbb{R})$ are **geodesics**!*

# Dirichlet-Voronoi fundamental domains in general

$(M, g)$ complete, connected Riemannian $N$-manifold, $d$ geodesic distance and $W \leq \mathrm{Isom}(M, g)$ *discrete* and $x_0 \in M$ regular point. The **Dirichlet-Voronoi** domain (centered at $x_0$) is

$$\mathcal{DV} := \{x \in M \; ; \; d(x_0, x) \leq d(wx_0, x), \; \forall w \in W\},$$

the $w$-**dissecting hypersurface** is $H_w := \{d(x_0, x) = d(wx_0, x)\}$.

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## Proposition

- *$\mathcal{DV}$ is a star-shaped fundamental domain for $W \circlearrowright M$.*
- *If $\mathcal{DV} \subset B_g(x_0, \rho)$ for $0 < \rho < \mathrm{inj}_{x_0}(M)$, where $\mathrm{inj}_{x_0}(M)$ is the **injectivity radius** of $M$ at $x_0$, then $\mathring{\mathcal{DV}}$ is a $N$-cell. Moreover in this case, we have a homeomorphism $\partial\mathcal{DV} \simeq \mathbb{S}^{N-1}$.*

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We hope to build an equivariant cell structure from $\mathcal{DV}$, where the lower cells should be intersections of **walls** $\mathcal{DV} \cap H_w$.

# The case of flag manifolds: first result

In general, $K/T$ admits a **normal homogeneous metric** (i.e. coming from a bi-invariant one on $K$, e.g. induced by the Killing form $\kappa$). We consider the Dirichlet-Voronoi domain

$$\mathcal{DV} := \{x \in K/T \ ; \ d(1,x) \leq d(w,x), \ \forall w \in W\}.$$

## Example

For $\mathcal{F}_n(\mathbb{C}) := SU(n)/S(U(1)^n)$ and $X, Y \in \mathfrak{su}(n)$, we have $\kappa(X,Y) = 2n\mathrm{tr}\,(XY)$.

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**Lemma**

- $\mathrm{inj}(\mathcal{F}_n(\mathbb{C}), \kappa) \geq \pi\sqrt{n/2}$,
- $\mathrm{inj}(\mathcal{F}_n(\mathbb{R}), \kappa) = \pi\sqrt{n} = d(1, s_\alpha)$ *for any* $\alpha \in \Phi^+$.

# A new structure on $\mathcal{F}_3(\mathbb{R})$

## Proposition

*Let* $\mathcal{DV}_3 \subset \mathcal{F}_3(\mathbb{R})$ *Dirichlet-Voronoi domain. Then*

$$\max_{x \in \mathcal{DV}_3} d(1,x) = 4\sqrt{3}\,\mathrm{arccos}(1/2+\sqrt{2}/4) \approx 3.7969 < 5.4414 \approx \pi\sqrt{3} = \mathrm{inj}(\mathcal{F}_3(\mathbb{R})).$$

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## Theorem (B3)

*The walls of $\mathcal{DV}_3$ induce an $\mathfrak{S}_3$-cell structure on $\mathcal{F}_3(\mathbb{R})$ with chain complex of the form $\mathbb{Z}[\mathfrak{S}_3] \to \mathbb{Z}[\mathfrak{S}_3]^7 \to \mathbb{Z}[\mathfrak{S}_3]^{12} \to \mathbb{Z}[\mathfrak{S}_3]^6$.*

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$$\max_{x \in \mathcal{DV}_3} d(1,x) = 4\sqrt{3}\arccos(1/2+\sqrt{2}/4) \approx 3.7969 < 5.4414 \approx \pi\sqrt{3} = \mathrm{inj}(\mathcal{F}_3(\mathbb{R})).$$

## Theorem (B3)

*The walls of $\mathcal{DV}_3$ induce an $\mathfrak{S}_3$-cell structure on $\mathcal{F}_3(\mathbb{R})$ with chain complex of the form $\mathbb{Z}[\mathfrak{S}_3] \to \mathbb{Z}[\mathfrak{S}_3]^7 \to \mathbb{Z}[\mathfrak{S}_3]^{12} \to \mathbb{Z}[\mathfrak{S}_3]^6$.*

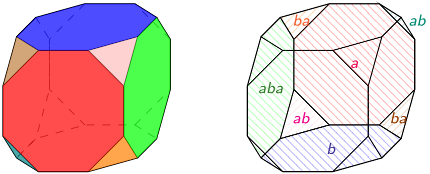


Figure: The 2-cells of $\mathcal{F}_3(\mathbb{R})$

# Summarizing

| **Problem A:** Maximal tori | **Problem B:** Flag manifolds |
|---|---|
| **Theorem A1:** *Equivariant triangulation of $T < K$ and dg-ring in the case $\pi_1(K) = 1$.* | **Theorem B1:** *Equivariant cell structure on $\mathcal{F}_3(\mathbb{R}) := SO(3)/S(O(1)^3)$ using $\mathbb{P}(\overline{\mathcal{O}_{\min}})$ and the GKM graph.* |
| **Theorem A2:** *Equivariant triangulation of $T < K$ in the general case using barycentric subdivision of alcoves.* | **Theorem B2:** *Equivariant cell structure on $\mathcal{F}_3(\mathbb{R})$ from the binary octahedral group $\mathcal{O} < \mathbb{S}^3$ of order 48.* |
| **Theorem A3:** *Construction of a $W$-triangulated analogue of tori for all finite irreducible Coxeter groups.* | **Theorem B3:** *Equivariant cell structure on $\mathcal{F}_3(\mathbb{R})$ from a normal homogeneous metric and a Dirichlet-Voronoi fundamental domain.* |

# Perspectives

- Work in progress: cell structure on $\mathcal{F}_n(\mathbb{R})$ using the domain $\mathcal{DV}$. Then extend to $\mathcal{F}_n(\mathbb{C})$ and to other types.
- $G \circlearrowright \mathfrak{g} \supset \mathcal{N}$ nilpotent cone and $\widetilde{\mathcal{N}} := \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \ ; \ x \in \mathfrak{b}\}$. Springer resolution $\pi : \widetilde{\mathcal{N}} \twoheadrightarrow \mathcal{N}$ and Springer fibers $\mathcal{B}_x := \pi^{-1}(x)$. $W$ acts on $R\Gamma(\mathcal{B}_x, \mathbb{Q})$ but not on $\mathcal{B}_x$ itself. $\mathcal{B}_0 = G/B$ is an actual $W$-space. Springer theory was a motivation for Problem B.
- The étale case of tori? Take the Frobenius into account!
- If $W$ complex reflection group, possible to construct a (compact) $W$-manifold generalizing $\mathbf{T}(W)$?
- Equivariant cell structure for $W \circlearrowright B_T \simeq (\mathbb{CP}^\infty)^n$? Geometric meaning of Koszul duality between $H^\bullet(B_T, \mathbb{Q}) = S^\bullet(\mathfrak{t})$ and $H^\bullet(T, \mathbb{Q}) = \Lambda^\bullet(\mathfrak{t}^*)$?

# Thank you !