Equivariant cellular structures on the flag manifold of $\mathbb{R}^3$ and Dirichlet-Voronoi domains

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Menu

$R\Gamma(X,\mathbb{Z}) \in \mathcal{D}^b(\mathbb{Z}[W])$ and equivariant cellular structures

discrete group $W \circlearrowright X$ topological space

$$\rightsquigarrow \; W \circlearrowright H^*(X,\mathbb{Z}) = H^*(R\Gamma(X,\mathbb{Z})).$$

Also, $R\Gamma(X,\mathbb{Z}) \in \mathcal{D}^b(\mathbb{Z}[W])$, but how to compute $R\Gamma(X,\mathbb{Z})$?

Definition

A CW-structure on X is W-**equivariant** if

- W acts on cells
- For $e \subset X$ a cell and $w \in W$, if $we = e$ then $w_{|e} = \mathrm{id}_e$.

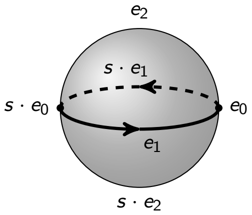
Associated **cellular chain complex**: $C_*^{\mathrm{cell}}(X, W; \mathbb{Z}) \in \mathcal{C}_b(\mathbb{Z}[W])$.

Theorem

The complex $C^*_{\mathrm{cell}}(X, W; \mathbb{Z})$ is well-defined up to homotopy and $C^*_{\mathrm{cell}}(X, W; \mathbb{Z}) \cong R\Gamma(X,\mathbb{Z})$ in $\mathcal{D}^b(\mathbb{Z}[W])$.

Illustration: $\{\pm 1\} \circlearrowright \mathbb{S}^2 \subset \mathbb{R}^3$

$C_2 = \{1, s\}$ acts on $\mathbb{S}^2$ via the antipode $s : x \mapsto -x$. We construct a C_2-equivariant cellular structure as follows:



Cochain complex given by

$$C^*_{\mathrm{cell}}(\mathbb{S}^2, C_2; \mathbb{Z}) = \left(\mathbb{Z}[C_2] \langle e_0^* \rangle \xrightarrow{1-s} \mathbb{Z}[C_2] \langle e_1^* \rangle \xrightarrow{1+s} \mathbb{Z}[C_2] \langle e_2^* \rangle \right)$$

so $R\Gamma(\mathbb{S}^2, \underline{\mathbb{Q}}) \simeq \mathbb{1} \oplus \varepsilon[-2]$ and $R\Gamma(\mathbb{S}^2, \underline{\mathbb{F}_2})$ has cohomology $H^*(\mathbb{S}^2, \mathbb{F}_2) = \mathbb{1} \oplus \mathbb{1}[-2]$, however, $R\Gamma(\mathbb{S}^2, \underline{\mathbb{F}_2})$ is indecomposable...

Flag manifolds

Notation

- G a connected reductive complex algebraic group,
- B a Borel subgroup of G, $T^{\mathbb{C}} < G$ maximal torus such that $T^{\mathbb{C}} < B$.
- $W := N_G(T^{\mathbb{C}})/T^{\mathbb{C}}$ the **Weyl group**.
- **Flag manifold**: the homogeneous space $\mathcal{F}_G(\mathbb{C}) := G/B$.
- If K maximal compact subgroup of G and $T := K \cap T^{\mathbb{C}} \simeq (\mathbb{S}^1)^r$, then

$$\mathcal{F}_G(\mathbb{C}) := G/B \overset{\text{diff}}{\simeq} K/T \circlearrowleft W = N_K(T)/T.$$

Problem

Describe $G/B = K/T$ as a W-equivariant CW-complex.

Example: type A

- $G = SL_n(\mathbb{C})$, $K = SU(n)$, $T = S(U(1)^n)$ and $W = \mathfrak{S}_n$,
- $B = \begin{pmatrix} * & * & \dots & * \\ 0 & * & & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & * \end{pmatrix}$ Borel subgroup,
- $G/B = \{\text{flags } (0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = \mathbb{C}^n)\}$,
- $K/T = \{\text{decompositions } \mathbb{C}^n = L_1 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} L_n\} \circlearrowleft \mathfrak{S}_n$,
- $G/T^{\mathbb{C}} = \{\text{decompositions } \mathbb{C}^n = L_1 \oplus \cdots \oplus L_n\} \circlearrowleft \mathfrak{S}_n$.

Example

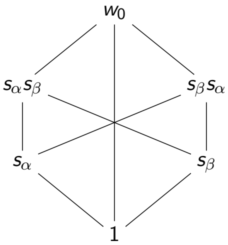
$$\mathcal{F}_{SL_2}(\mathbb{C}) \simeq U(2)/U(1)^2 \simeq \mathbb{CP}^1 \circlearrowleft \mathfrak{S}_2 = \langle s \rangle,\ s(L_1 \overset{\perp}{\oplus} L_2) = L_2 \overset{\perp}{\oplus} L_1.$$

$$[1 : z] \cdot s = [-\overline{z} : 1] = [1 : -1/\overline{z}]$$

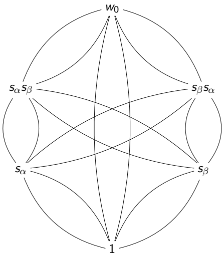
$\rightsquigarrow$ antipode on $\mathbb{S}^2$, as in the example above.

The Goresky-Kottwitz-MacPherson graph

$\{\text{vertices}\} \leftrightarrow \{T\text{-fixed points}\}$, $\{\text{edges}\} \leftrightarrow \{T\text{-orbits of dim 1}\}$, each edge $\leftrightarrow \overline{T\text{-orbit}} = \mathbb{CP}^1$



(a) The GKM graph



(b) Many SL_2 situations

$\rightsquigarrow$ 1-skeleton and part of the 2-skeleton, like for SL_2.

Educated guess/hope

- Hope the complex has nice general combinatorial description.
- Hard to find a CW-structure. Try to guess ranks of free modules. Let

$$P_W^{\mathbb{C}}(q) := \sum_k \#\{W\text{-orbits of } k\text{-cells of } \mathcal{F}\ell_{\mathbb{C}}(G)\} q^k$$

and similarly consider $P_W^{\mathbb{R}}$ for the real points.
Constraints: $\deg(P_W^{\mathbb{C}}) = 2|\Phi^+|$ and $P_W^{\mathbb{C}}(-1) = 1$.

- Parametrization

$$\{k\text{-cells of } \mathcal{F}\ell_{\mathbb{R}}(G)\} \leftrightarrow \{k\text{-subsets of positive roots}\}.$$

k-cells parametrized by k real parameters (one for each root).
This would give $P_W^{\mathbb{R}}(q) = [2]_q^{|\Phi^+|}$.
Recall that $[k]_q = 1 + q + \cdots + q^{k-1}$.

Educated guess/hope

- Missing cells in $\mathcal{F}\ell_{\mathbb{C}}(G)$: allow some parameters to take complex values.

  Compatible with GKM.

  Each positive root would have a multiplicity 0, 1 or 2

  $\rightsquigarrow$ multiset of positive roots with multiplicity.

  This gives $P_W^{\mathbb{C}}(q) = [3]_q^{|\Phi^+|}$.

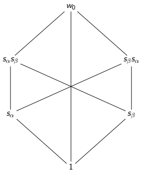
  $\rightsquigarrow$ combinatorial flavour of the de Concini–Salvetti complex.

  Recall that the de Concini–Salvetti complex is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[W]$, with W finite Coxeter group, constructed using increasing chains of subsets of simple reflections.

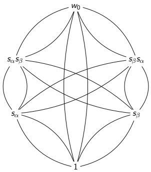
  For SL_3, $P_W^{\mathbb{R}}(q) = [2]_q^3 = q^3 + 3q^2 + 3q + 1$ and $P_W^{\mathbb{C}}(q) = [3]_q^3 = q^6 + 3q^5 + 6q^4 + 7q^3 + 6q^2 + 3q + 1$.

A first decomposition of $\mathcal{F}_3(\mathbb{R}) := \mathcal{F}_{SL_3}(\mathbb{R}) = SL_3(\mathbb{R})/B$

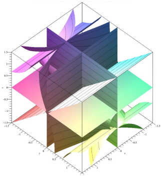
$\mathcal{O}_{\min} := SL_3(\mathbb{C}) \cdot E_{1,3}$ **minimal nilpotent orbit**, then $\mathcal{F}_3(\mathbb{C}) = \mathbb{P}(\overline{\mathcal{O}_{\min}}) \subset \mathbb{P}(\mathfrak{sl}_3) \simeq \mathbb{CP}^7$, so $\mathcal{F}_3(\mathbb{R}) \hookrightarrow \mathbb{RP}^7$. Using the **GKM graph** of $\mathfrak{S}_3$, construct an equivariant structure on $\mathcal{F}_3(\mathbb{R})$.



(a) GKM graph



(b) SL_2 situations



(c) 3-cells of $\mathcal{F}_3(\mathbb{R})$

Theorem (G. 2019)

- *$\mathcal{F}_3(\mathbb{R})$ admits an $\mathfrak{S}_3$-cellular structure whose cellular chain complex has the shape $\mathbb{Z}[\mathfrak{S}_3]^4 \to \mathbb{Z}[\mathfrak{S}_3]^6 \to \mathbb{Z}[\mathfrak{S}_3]^3 \to \mathbb{Z}[\mathfrak{S}_3]$.*
- *There is an $\mathfrak{S}_3$-isomorphism $\mathbb{F}_2[x, y, z]_{\mathfrak{S}_3} \to H^*(\mathcal{F}_3(\mathbb{R}), \mathbb{F}_2)$ sending x, y and z to irreducible real algebraic 1-cocycles.*

New look at $\mathcal{F}_3(\mathbb{R})$

Recall $\mathcal{F}_3(\mathbb{R}) = SO(3)/S(O(1)^3) = SO(3)/\{\pm 1\}^2 \circlearrowleft \mathfrak{S}_3$.

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[d, "/\{\pm 1\}"] \arrow[dd, bend right, "/\mathcal{Q}_8"'] & \circlearrowleft \mathcal{Q}_8 \rtimes \mathfrak{S}_3 = \mathcal{O} = \left\langle i, \frac{1}{\sqrt{2}}(1+j) \right\rangle \\
SO(3) \arrow[d, "/\{\pm 1\}^2"] & \circlearrowleft \{\pm 1\}^2 \rtimes \mathfrak{S}_3 = W(D_3) = \mathfrak{S}_4 \\
SO(3)/\{\pm 1\}^2 \arrow[d, "/\mathfrak{S}_3"] & \circlearrowleft \mathfrak{S}_3 \\
\mathbb{S}^3/\mathcal{O} & \circlearrowleft 1
\end{tikzcd}$$

where $\mathcal{O}$ is the **binary octahedral group**. This last space is called a **spherical space form**. Construct an $\mathcal{O}$-cellular structure on $\mathbb{S}^3$?

Orbit polytopes

Free isometric action of a finite group $\mathcal{G} \circlearrowright \mathbb{S}^n \ni v_0$.

The **orbit polytope** of $\mathcal{G}$ is $P_\mathcal{G} := \mathrm{conv}(\mathcal{G} \cdot v_0)$.

$\mathcal{G}$ acts freely on $P_\mathcal{G}$ and on its faces and the projection

$$\partial P_\mathcal{G} \to \mathbb{S}^n$$

is a $\mathcal{G}$-homeomorphism.

Theorem (*Fêmina–Galves–Manzoli Neto–Spreafico (2013), Chirivì–Spreafico (2017)*)

Assume that $\mathrm{span}(\mathcal{G} \cdot v_0) = \mathbb{R}^{n+1}$. *Then there is a system* $F_1, \ldots, F_r$ *of orbit representatives for the* $\mathcal{G}$*-action on facets of* $P_\mathcal{G}$ *such that* $\bigcup_i F_i$ *is a fundamental domain for* $\mathcal{G}$ *on* $\partial P_\mathcal{G}$.

The chain complex

Theorem (*Chirivì–G.–Spreafico, 2020*)

The sphere $\mathbb{S}^3$ admits an $\mathcal{O}$-equivariant cellular decomposition whose associated cellular homology complex is

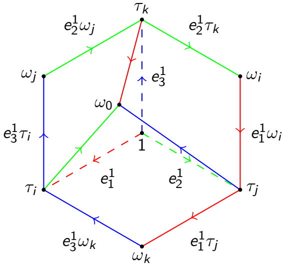
$$\mathbb{Z}[\mathcal{O}] \xrightarrow[\partial_3]{\begin{pmatrix} 1-\tau_i & 1-\tau_j & 1-\tau_k \end{pmatrix}} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow[\partial_2]{\begin{pmatrix} \omega_i & 1 & \tau_j - 1 \\ \tau_k - 1 & \omega_j & 1 \\ 1 & \tau_i - 1 & \omega_k \end{pmatrix}} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow[\partial_1]{\begin{pmatrix} \tau_i - 1 \\ \tau_j - 1 \\ \tau_k - 1 \end{pmatrix}} \mathbb{Z}[\mathcal{O}] \ ,$$

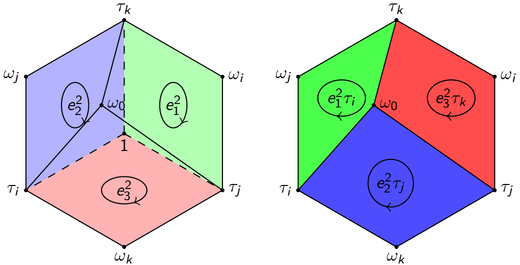
where

$$\omega_0 = \frac{1-i-j-k}{2},\ \omega_i = \frac{1+i-j-k}{2},\ \omega_j = \frac{1-i+j-k}{2},\ \omega_k = \frac{1-i-j+k}{2},$$

$$\tau_i = \frac{1-i}{\sqrt{2}},\ \tau_j = \frac{1-j}{\sqrt{2}},\ \tau_k = \frac{1-k}{\sqrt{2}}.$$

What does it look like?





The complex $1, 3, 3, 1$ for $\mathcal{F}_3(\mathbb{R})$

Theorem

The real flag manifold $\mathcal{F}_3(\mathbb{R}) \simeq \mathbb{S}^3/\mathcal{Q}_8$ admits an $\mathfrak{S}_3$-equivariant cell structure with cellular chain complex given by

$$\mathbb{Z}[\mathfrak{S}_3] \xrightarrow[\partial_3]{\begin{pmatrix} 1-s_\beta & 1-w_0 & 1-s_\alpha \end{pmatrix}} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow[\partial_2]{\begin{pmatrix} s_\alpha s_\beta & 1 & w_0-1 \\ s_\alpha - 1 & s_\alpha s_\beta & 1 \\ 1 & s_\beta - 1 & s_\alpha s_\beta \end{pmatrix}} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow[\partial_1]{\begin{pmatrix} 1-s_\beta \\ 1-w_0 \\ 1-s_\alpha \end{pmatrix}} \mathbb{Z}[\mathfrak{S}_3],$$

$w_0 = s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$ *being the longest element of* $\mathfrak{S}_3 = \langle s_\alpha, s_\beta \rangle$.

Remark

The 1-cells of $\mathbb{S}^3$ and $\mathcal{F}_3(\mathbb{R})$ are ***geodesics****!...*

Dirichlet-Voronoi fundamental domains in general

(M, g) complete, connected Riemannian N-manifold, d geodesic distance and $W \leq \mathrm{Isom}(M, g)$ *discrete* and $x_0 \in M$ regular point. The **Dirichlet-Voronoi** domain (centered at x_0) is

$$\mathcal{DV} := \{x \in M \ ; \ d(x_0, x) \leq d(wx_0, x), \ \forall w \in W\},$$

the w-**dissecting hypersurface** is $H_w := \{d(x_0, x) = d(wx_0, x)\}$.

Proposition

- *$\mathcal{DV}$ is a star-shaped fundamental domain for $W \circlearrowright M$.*
- *If $\mathcal{DV} \subset B_g(x_0, \rho)$ for $0 < \rho < \mathrm{inj}_{x_0}(M)$, where $\mathrm{inj}_{x_0}(M)$ is the **injectivity radius** of M at x_0, then $\mathring{\mathcal{DV}}$ is a N-cell. Moreover in this case, we have a homeomorphism $\partial\mathcal{DV} \simeq \mathbb{S}^{N-1}$.*

We hope to build an equivariant cell structure from $\mathcal{DV}$, where the lower cells should be intersections of **walls** $\mathcal{DV} \cap H_w$.

Injectivity radius

Recall that

$$\begin{aligned}\mathrm{inj}_{x_0}(M) &= \sup\{r > 0 \text{ s.t. } \mathrm{Exp}_{x_0} \text{ is injective on } B(x_0, r)\} \\ &= \sup\{r > 0 \; ; \; \forall x \in B(x_0, r), \; \exists! \text{ minimal geodesic from } x_0 \text{ to } x\}.\end{aligned}$$

Hard to compute! but some estimates, e.g. using **sectional curvature** (Klingenberg, '59).

The case of flag manifolds: first result

In general, K/T admits a **normal homogeneous metric** (i.e. coming from a bi-invariant one on K, e.g. induced by the Killing form κ). We consider the Dirichlet-Voronoi domain

$$\mathcal{DV} := \{x \in K/T \ ; \ d(1,x) \leq d(w,x), \ \forall w \in W\}.$$

Example

For $\mathcal{F}_n(\mathbb{C}) := SU(n)/S(U(1)^n)$ and $X, Y \in \mathfrak{su}(n)$, we have $\kappa(X,Y) = 2n\mathrm{tr}\,(XY)$.

Conjecture: $\mathcal{DV} \subset B(1, \mathrm{inj}(K/T))$. First step:

Lemma

We have $\mathrm{inj}(\mathcal{F}_n(\mathbb{C}), \kappa) \geq \pi\sqrt{n/2}$ *and* $\mathrm{inj}(\mathcal{F}_n(\mathbb{R}), \kappa) = \pi\sqrt{n} = d(1, s_\alpha)$ *for any* $\alpha \in \Phi^+$.

A new structure on $\mathcal{F}_3(\mathbb{R})$

Proposition

Let $\mathcal{DV}_3 \subset \mathcal{F}_3(\mathbb{R})$ Dirichlet-Voronoi domain. Then

$$\max_{x \in \mathcal{DV}_3} d(1,x) = 4\sqrt{3}\arccos(1/2+\sqrt{2}/4) \approx 3.7969 < 5.4414 \approx \pi\sqrt{3} = \mathrm{inj}(\mathcal{F}_3(\mathbb{R})).$$

Theorem (G. 2021)

The walls of $\mathcal{DV}_3$ induce an $\mathfrak{S}_3$-cell structure on $\mathcal{F}_3(\mathbb{R})$ with chain complex of the form $\mathbb{Z}[\mathfrak{S}_3] \to \mathbb{Z}[\mathfrak{S}_3]^7 \to \mathbb{Z}[\mathfrak{S}_3]^{12} \to \mathbb{Z}[\mathfrak{S}_3]^6$.

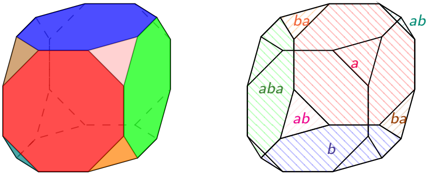


Figure: The 2-cells of $\mathcal{F}_3(\mathbb{R})$

A glance at the proof of $\max_{\mathcal{DV}_3} d(1,-) \approx 3.7969$

Lemma ("no antenna lemma")

(M,g) connected and compact, $W \leq \mathrm{Isom}(M)$ finite, $x_0 \in M$ regular and assume that $\mathcal{DV} \subset B(x_0, \mathrm{inj}_{x_0}(M))$. If $\mathcal{DV} \cap S(x_0,\rho)$ is finite for some $0 < \rho < \mathrm{inj}_{x_0}(M)$, then $\mathcal{DV} \subset \overline{B(x_0,\rho)}$. If moreover $\mathcal{DV} \cap S(x_0,\rho) \neq \emptyset$ then $\max_{x\in\mathcal{DV}} d(x_0,x) = \rho$.

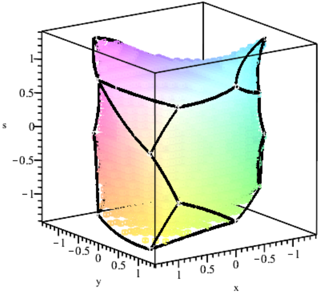
Let $\pi : \mathbb{S}^3 \twoheadrightarrow \mathcal{F}_3(\mathbb{R})$. For $q = a + bi + cj + dk \in \mathbb{S}^3$ s.t. $a \geq |b|,|c|,|d|$, we have $d(1,\pi(q)) = 4\sqrt{3}\arccos(a)$. Using reflections, prove that $\pi(q) \in \mathcal{DV}_3 \;\Rightarrow\; a^2(10-6\sqrt{2}) \geq 1$, so $\mathcal{DV}_3 \subset B(1,\pi\sqrt{3})$. Let $a_0 := 1/2 + \sqrt{2}/4$, conditions on remaining elements of $\mathfrak{S}_3$ yields

$$\{\pi(q) \in \mathcal{DV}_3 \;;\; d(1,\pi(q)) = 4\sqrt{3}a_0\}$$

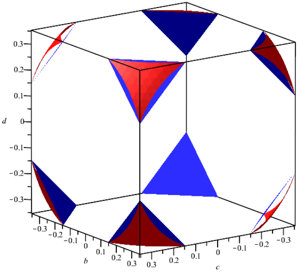
$$\approx \{(b,c,d) \in \mathbb{R}^3 \;;\; |b|,|c|,|d| \leq \sqrt{2}/4,\; |b \pm c \pm d| \leq a_0,\; b^2+c^2+d^2 = 1 - a_0^2\}$$

$\rightsquigarrow$ full truncated cube intersected with a sphere $\rightsquigarrow$ only 24 points (the 0-cells). $\square$

Thank you!



(a) Other view of $\mathcal{DV}_3$



(b) Truncated cube (blue) and sphere (red)

The complex for the first decomposition is

$$\mathbb{Z}[\mathfrak{S}_3]^4 \xrightarrow{\partial_3} \mathbb{Z}[\mathfrak{S}_3]^6 \xrightarrow{\partial_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{\partial_1} \mathbb{Z}[\mathfrak{S}_3] \ ,$$

where

$$\partial_1 = \begin{pmatrix} 1-s_\alpha & 1-s_\beta & 1-w_0 \end{pmatrix}, \quad \partial_3 = \begin{pmatrix} 0 & s_\alpha & 0 & 1 \\ -s_\beta s_\alpha & 0 & -w_0 & 0 \\ 0 & s_\beta s_\alpha & 1 & 0 \\ 1 & 0 & 0 & s_\beta s_\alpha \\ -s_\alpha s_\beta & s_\alpha s_\beta & 0 & 0 \\ 0 & 0 & s_\alpha s_\beta & -s_\alpha s_\beta \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} -1 & 1 & 1 & s_\alpha & w_0 - s_\alpha s_\beta & s_\beta - s_\beta s_\alpha \\ s_\beta s_\alpha - s_\beta & s_\alpha - 1 & -w_0 & w_0 & s_\alpha s_\beta & s_\alpha s_\beta \\ s_\beta & s_\beta s_\alpha & s_\alpha - 1 & s_\alpha s_\beta - w_0 & -s_\beta & s_\beta s_\alpha \end{pmatrix}.$$

The complex for the third decomposition is

$$\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{\partial_3} \mathbb{Z}[\mathfrak{S}_3]^7 \xrightarrow{\partial_2} \mathbb{Z}[\mathfrak{S}_3]^{12} \xrightarrow{\partial_1} \mathbb{Z}[\mathfrak{S}_3] ,$$

where

$$\partial_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & s_\beta & -s_\beta & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & s_\beta s_\alpha & 0 & 0 & -1 \\ -w_0 & 0 & 0 & 0 & 0 & 0 & 0 & w_0 & 0 & s_\beta & -w_0 & 0 \\ s_\beta s_\alpha & -s_\beta s_\alpha & 0 & 0 & s_\alpha & -s_\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_\beta s_\alpha & -s_\beta s_\alpha & 0 & 0 & 0 & 0 & 0 & -w_0 & 0 & w_0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & s_\beta & -s_\beta & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} 1 & 0 & w_0 & 0 & 0 & 0 & -w_0 \\ 1 & -s_\alpha s_\beta & 0 & 0 & -1 & 0 & 0 \\ 1 & s_\beta & 0 & -s_\beta & 0 & 0 & 0 \\ 1 & 0 & s_\alpha & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -w_0 & 0 & 0 \\ 1 & s_\alpha & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -s_\beta & 0 \\ 1 & 0 & -s_\beta s_\alpha & -1 & 0 & 0 & 0 \\ 0 & -1 & -w_0 & 0 & -s_\beta & 0 & 0 \\ 0 & s_\beta & 1 & 0 & 0 & 0 & -s_\beta \\ 0 & w_0 & -w_0 & -1 & 0 & 0 & 0 \\ 0 & -s_\beta s_\alpha & 1 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \partial_3 := \begin{pmatrix} 1-s_\alpha \\ 1-s_\beta \\ 1-w_0 \\ 1-s_\beta s_\alpha \\ 1-s_\alpha s_\beta \\ 1-s_\beta s_\alpha \\ 1-s_\alpha s_\beta \end{pmatrix}.$$