Enumerative geometry & Schubert calculus

From the four lines to cohomology and characteristic classes

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Basic idea

Question

Enumerate the number of solutions of a geometric problem ?

Example

How many lines intersect four given lines in 'general position' ?

Answer : Two. But how do we do this ?

Basic idea

Hermann Schubert found a way of doing such calculations with symbolic notations. The letters represent some kind of condition and we doo 'boolean' calculus with these notations.

In a 'configuration space', if x denotes a condition on elements of this space and y denotes another condition, then $x + y$ is the condition that x OR y is satisfied, and $x \cdot y$ is the condition that x AND y are satisfied.

Basic idea

In this way, one can write a given condition in terms of 'basic conditions' and eventually get the number of solutions to the problem given in the first place.

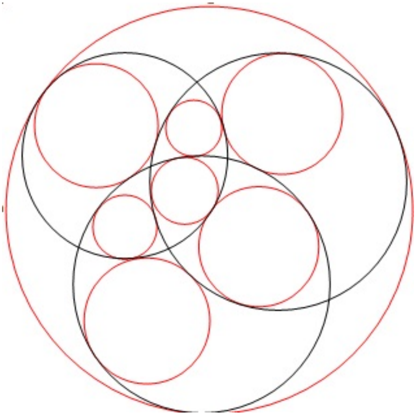
This is called Schubert calculus.

Gives amazing results : There are 3264 conics that are tangent to 5 given conics !

Rigorous foundations of this ? Hilbert's problem 15 !

Examples

Another example of enumerative geometry : the circles of Apollonius...



...and another one

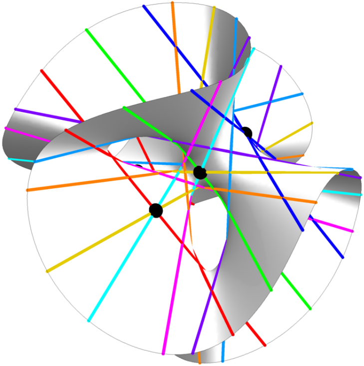


Figure – The 27 lines contained in the Clebsch cubic

Warm up

Consider the following conditions on points of $\mathbb{P}^3$:

Notation	Condition	Dimension
p	the point lies on a given plane	2
p_g	the point lies in a given line	1
P	the point itself is given	0

We have the formula

$$p^2 = p_g.$$

But for this, the planes must be in 'general position'. We note a fruitful ambiguity here...

Warm up

In the same way, we get the following formulae :

$$p^3 = pp_g, \quad pp_g = P, \quad p^3 = P.$$

Setting the stage

Consider the Grassmann variety $\mathcal{G} := \mathbb{G}(1,3)$. This is the set of lines in $\mathbb{P}^3$ and it is a smooth projective variety over $\mathbb{C}$. It also can be seen as the set $G(2,4)$ of 2-planes of $\mathbb{C}^4$.

Choose a *flag* $P \subset g \subset e \subset \mathbb{P}^3$.

Basic conditions on lines :

Notation	Condition	Dimension
g	the line cuts a given line	3
g_e	the line lies in a given plane	2
g_p	the line contains a given point	2
g_s	the line belongs to a given pencil	1
G	the line itself is given	0

Let Ω_g (resp. Ω_e, Ω_p, Ω_s and Ω_G) the set of lines satisfying condition g (resp. g_e, g_p, g_s and G).

Setting the stage

We have the following diagram of inclusions

$$\begin{array}{ccccccc} & & & \Omega_p & & & \\ & & \nearrow & & \searrow & & \\ \Omega_G & \longrightarrow & \Omega_s & & & \Omega_g & \longrightarrow & \mathcal{G} \\ & & \searrow & & \nearrow & & \\ & & & \Omega_e & & & \end{array}$$

and this is a (minimal) cellular decomposition of $\mathcal{G}$.

Setting the stage

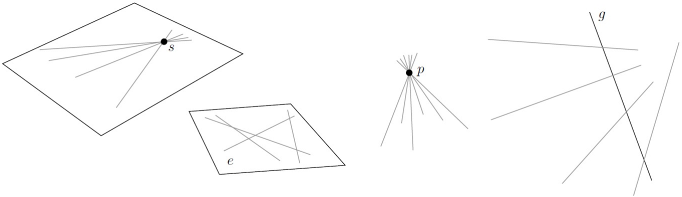


Figure – Schubert cycles Ω_s, Ω_e, Ω_p and Ω_g

The four lines problem 'solved'

Now we can solve the four lines problem in Schubert's way :

First write g^2 in terms of basic conditions.

Take g and g' two lines in general positions. With the notations above, one has

$$\Omega_g \cap \Omega_{g'} = \Omega_p \cup \Omega_e.$$

Then, we get (since intersections are transversal) :

$$g^2 = g_e + g_p. \tag{1}$$

The four lines problem 'solved'

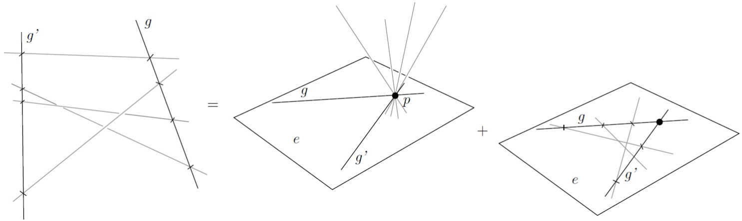


Figure – How to see that $g^2 = g_e + g_p$

The four lines problem 'solved'

In the same fashion, one can prove the following formulae :

$$gg_p = g_s, \quad gg_e = g_s, \quad gg_s = G.$$

Multiplying (1) by g, we get

$$g^3 = gg_e + gg_p = 2g_s,$$

Multiplyging by g again, we obtain

$$g^4 = 2gg_s = 2G.$$

This means that there are 2 lines that cut two given lines in $\mathbb{P}^3$.

Prinzip der Erhaltung der Anzahl

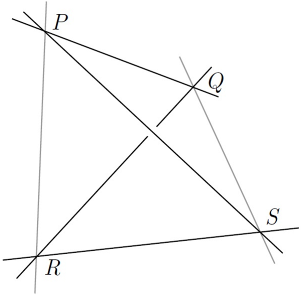
There is a fundamental principle in Schubert calculus, called the *Principle of conservation of number*.

It says that the number of solutions to an enumerative problem remains unchanged when the parameters vary, provided that the number of solutions is still finite...

... a bit like the number of roots of a (one variable) polynomial is unchanged when the coefficients vary, provided that it doesn't become zero.

Prinzip der Erhaltung der Anzahl

From this principle, we can solve the four lines problem using tangent spaces and transversal intersections... We apply it to the following moderately degenerate case :



Algebraic topology flavour

Interesting remark about Schubert's method :

Remark

If X and Y are 'condition loci' in a configuration space (like $\Omega_g \subset \mathcal{G}$ above for instance) and if $[X]$ and $[Y]$ denote the symbolic notation for conditions, then one has (in general position)

$$[X] \cdot [Y] = [X \cap Y].$$

This lets us guess that cohomology and intersection theory have something to do with it !

Reminder on singular cohomology

Example

Let $X \in \mathfrak{Top}$ be a topological space. Take, for $n \geq 0$, $C_n(X, \mathbb{Z})$ to be the free abelian group generated by *singular simplicies* $\sigma : \Delta^n \to X$.

These can be turned into a *chain complex* using differentials $\partial_n : C_n(X, \mathbb{Z}) \to C_{n-1}(X, \mathbb{Z})$ and the homology of this, written $H_*(X, \mathbb{Z})$, is the *singular homology*.

Applying the Hom functor to $C_*(X, \mathbb{Z})$, we get a *cochain complex* $C^*(X, \mathbb{Z}) = \mathrm{Hom}\left(C_*(X, \mathbb{Z}), \mathbb{Z}\right)$ whose cohomology is the *singular cohomology* $H^*(X, \mathbb{Z})$ of X. This gives a functor

$$H^*(-, \mathbb{Z}) : \mathfrak{Top} \to \mathfrak{Ab}^{\text{graded}}.$$

Reminder on singular cohomology

Definition

There is a product

$$- \cup - : H^n(X,\mathbb{Z}) \times H^m(X,\mathbb{Z}) \to H^{n+m}(X,\mathbb{Z}),$$

called the *cup product*.

This makes $H^*(X,\mathbb{Z})$ into a graded $\mathbb{Z}$-algebra. This additional law gives the cohomology a richer structure and it's a finer invariant than cohomology seen as a graded abelian group.

Thus, we get a functor

$$H^*(-,\mathbb{Z}) : \mathfrak{Top} \to \mathfrak{Alg}_{\mathbb{Z}}^{\text{graded}}.$$

Poincaré duality

Question

Is there a link between homology and cohomology ?

In fact, if M is a real $\mathcal{C}^\infty$-manifold of dimension n, under certain technical hypothesis (closed, compact and orientable), then one has the *Poincaré duality* : there are isomorphisms

$$\forall 0 \leq k \leq n,\ H^k(M,\mathbb{Z}) \overset{\simeq}{\to} H_{n-k}(M,\mathbb{Z}).$$

In particular, this holds for complex projective algebraic varieties...

Fundamental class of a subvariety

Definition

If X is a smooth projective variety, and V is a closed irreducible subvariety, then there is a *fundamental class* $[V] \in H_{2\dim V}(X, \mathbb{Z})$.

This is a canonical generator of $H_{2\dim V}(V, \mathbb{Z})$.

Using Poincaré duality, we get a *dual class* $[V]^* \in H^{2c}(X, \mathbb{Z})$, where $c = \operatorname{codim}_X(V)$. This will be crucial in what follows.

Vector bundles and characteristic classes

Definition

Given an algebraic variety X, a *vector bundle* ξ of rank r on X is a triplet $\xi = (E, p, X)$ with E and p algebraic, p surjective and 'locally trivial', such that, for all $x \in X$, $p^{-1}(x)$ is a $\mathbb{C}$-vector space of dimension r.

Example

The *tautological bundle* τ_n on $\mathbb{P}^n$ of rank 1 : take $E := \{(\ell, x) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \; ; \; x \in \ell\}$ and $p := \mathrm{pr}_1 : (\ell, x) \mapsto \ell$.

Definition

A *section* of a vector bundle $\xi = (E, p, X)$ is a morphism $s : X \to E$ such that $p \circ s = id_X$.

Vector bundles and characteristic classes

Definition

To a vector bundle ξ of rank r on an algebraic variety X, one can associate *Chern classes* (that are characteristic classes) $c_i(\xi) \in H^{2i}(X, \mathbb{Z})$ for $0 \leq i \leq r$, characterized by the following properties :

- $c_0(\xi) = 1$,
- For $f : Y \to X$ continuous, we have $c_i(f^*(\xi)) = f^*(c_i(\xi))$,
- If $\xi = \xi' \oplus \xi''$, then $c_k(\xi) = \sum_{i+j=k} c_i(\xi') \cup c_j(\xi'')$,
- If τ_n is the tautological line bundle over $\mathbb{P}^n$, then $c_1(\tau_n) = [H]^*$, with $[H]^*$ the dual class of the hyperplane $\mathbb{P}^{n-1}$ and $c_i(\tau_n) = 0$ if $i > 1$.

Vector bundles and characteristic classes

Definition

Let ξ be a v.b. of rank r over X.

* The *Euler class* of ξ is $e(\xi) := c_r(\xi)$.
* The *Segre classes* of ξ are the classes $s_i(\xi) \in H^{2i}(X, \mathbb{Z})$ such that $s(\xi) := \sum_i s_i(\xi)$ is a formal inverse of the *total Chern class* $c(\xi) := \sum_i c_i(\xi)$.

The cohomology ring of $\mathcal{G} = \mathbb{G}(1,3)$

Let $\eta = (E, p, \mathcal{G})$ be the *tautological bundle* over $\mathcal{G}$, with $E = \{(\ell, x) \in \mathcal{G} \times \mathbb{C}^4 \ ; \ x \in \ell\}$ and $p = \mathrm{pr}_1$.

We have its Chern classes $c_i := c_i(\eta) \in H^{2i}(\mathcal{G}, \mathbb{Z})$ for $0 \leq i \leq 2$, and its Segre classes $s_j := s_j(\eta)$ for $0 \leq j \leq 4$, intertwined by the equation

$$(1 + c_1 + c_2)(1 + s_1 + s_2 + s_3 + s_4) = 1.$$

This gives

$$\begin{cases} s_1 = -c_1, \\ s_2 = c_1^2 - c_2, \\ s_3 = 2c_1c_2 - c_1^3, \\ s_4 = c_1^4 - 3c_1^2c_2 + c_2^2 \end{cases}.$$

The cohomology ring of $\mathcal{G}$

Theorem

We have an isomorphism of rings

$$H^*(\mathcal{G},\mathbb{Z}) \simeq \mathbb{Z}[c_1,c_2] \Big/ (c_1^3 - 2c_1c_2,\, c_2^2 - c_1^2c_2).$$

Cohomological translation of geometric conditions

Recall the notations g, g_e, g_p, g_s, G and the corresponding subvarieties $\Omega_g, \Omega_e, \Omega_p, \Omega_s, \Omega_G$. We aim to find the dual classes of these subvarieties, called the *Schubert (dual) classes*.

Let's investigate the case of Ω_g. Choose a basis (v_1, v_2, v_3, v_4) of $\mathbb{C}^4$, $g := \mathrm{Vect}(v_1, v_2)$ and $\widetilde{g}$ the associated trivial v.b. or rank 2 on $\mathcal{G}$. If $\varepsilon = \mathcal{G} \times \mathbb{C}^4$ is the trivial bundle, then $\eta \hookrightarrow \varepsilon$ and let

$$\varphi : \eta \hookrightarrow \varepsilon \twoheadrightarrow \varepsilon \big/ \widetilde{g}.$$

If $\Sigma(\varphi)$ is the *singular locus* of φ, i.e.

$$\Sigma(\varphi) := \left\{ \ell \in \mathcal{G} \ ; \ \ker\left(\varphi_{|\ell} : \ell \to \mathbb{C}^4 \big/ g \right) \neq 0 \right\},$$

then we see that

$$\Sigma(\varphi) = \Omega_g.$$

Cohomological translation of geometric conditions

If $\Lambda^2(\varphi)$ is the second exterior power of φ, then

$$\Lambda^2(\varphi) \in \mathrm{Hom}_{\,\mathrm{v.b.}} \left(\Lambda^2(\eta), \Lambda^2\left(\varepsilon \middle/ \widetilde{g}\right)\right)$$

$$\simeq \Gamma\left(\mathcal{G}, \Lambda^2(\eta)^* \otimes \Lambda^2\left(\varepsilon \middle/ \widetilde{g}\right)\right) \simeq \Gamma(\mathcal{G}, \Lambda^2(\eta)^*).$$

If $s_\varphi : \mathcal{G} \to E(\Lambda^2(\eta)^*)$ is the corresponding section, then one has

$$\Omega_g = \Sigma(\varphi) = s_\varphi^{-1}(0).$$

Cohomological translation of geometric conditions

Theorem

In this setting, one has

$$[s_\varphi^{-1}(0)]^* := \mathcal{PD}([s_\varphi^{-1}(0)]) = e(\Lambda^2(\eta)^*).$$

This is the link between geometry and cohomology that we need here !

Here, this finally leads to

$$X_g := [\Omega_g]^* = [\Sigma(\varphi)]^* = [s_\varphi^{-1}(0)]^* = e(\Lambda^2(\eta)^*)$$

$$= c_1(\Lambda^2(\eta)^*) = -c_1(\Lambda^2(\eta)) = -c_1(\eta) = s_1.$$

Cohomological translation of geometric conditions

In a similar fashion, we can compute the other Schubert classes and obtain the following recapitulation :

Symbolic notations	$-$	g	g_p	g_e	g_s	G
Schubert cycles	$\mathcal{G}$	Ω_g	Ω_p	Ω_e	Ω_s	Ω_G
Schubert classes	1	s_1	s_2	c_2	$s_1 c_2$	c_2^2

Finally, a girl is no one...

We recover the fact that two lines cut four given lines in general position : If the four lines $g_1, \ldots, g_4$ are in general position, that is they meet transversally, then we have that the cup product is Poincaré dual to intersection, i.e.

$$[\Omega_{g_1} \cap \cdots \cap \Omega_{g_4}]^* = [\Omega_{g_1}]^* \cup \cdots \cup [\Omega_{g_4}]^* = ([\Omega_g]^*)^4 = X_g^4$$

whence, by the previous computations,

$$X_g^4 = s_1^4 = c_1^4 = 2c_1^2 c_2 = 2c_2^2 = 2X_G$$

hence the result.

Flag variety of a Lie group

Definition

Recall that a Lie group K is a $\mathcal{C}^\infty$-manifold, endowed with the structure of a group, such that the map $K \times K \to K$, $(x, y) \mapsto xy^{-1}$ is a smooth map.

The *Lie algebra* of K is the tangent space $\mathfrak{k} := T_e K$.

Take K a compact connected *semisimple* Lie group, T a maximal torus (i.e. a maximal subgroup isomorphic to $(\mathbb{S}^1)^n$). Let G be a *complexification* of K, and B a Borel subgroup of G. By Iwasawa decomposition, one has a diffeomorphism

$$G/B \simeq K/T,$$

and this is called the *flag variety of K* (or G). This is a smooth projective variety.

Flag variety of a Lie group

Example

If $K = SU(n)$, then $T \simeq (\mathbb{S}^1)^n$, $G = SL_n(\mathbb{C})$ and we can take B to be the upper triangular matrix group. Then

$$G/B \simeq \mathcal{F}\ell,$$

where

$$\mathcal{F}\ell := \{(F_1, \dots, F_n) \ ; \ F_i \subset \mathbb{C}^n, \ F_i \subset F_{i+1}, \ \dim F_i = i\}.$$

The Weyl group and Schubert classes

Definition

With the above notations, let

$$W := N_K(T)/T.$$

It is a finite group, called the *Weyl group* of K. It is a Coxeter group.

It acts naturally on K/T, as well as on the *root system* of K.

Example

The Weyl group of $SU(n)$ is the symmetric group $\mathfrak{S}_n$.

The Weyl group and Schubert classes

Theorem

We have the Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB,$$

hence, we also have a decomposition

$$G/B = \bigsqcup_{w \in W} BwB/B.$$

Furthermore, we have an isomorphism of algebraic varieties

$$\forall w \in W,\ BwB/B \simeq \mathbb{C}^{\ell(w)}.$$

The Weyl group and Schubert classes

Definition

The (Zariski) closure of BwB/B in G/B, written $Y_w = \overline{BwB/B}$ and is called the *Schubert variety* associated to $w \in W$.

This is a closed irreducible subvariety of G/B, and if $X_w := [Y_w]^* \in H^{2(\mathrm{rk}(G)-\ell(w))}(G/B,\mathbb{Z})$ is the dual class, called a *Schubert class*, then

$$H^*(G/B,\mathbb{Z}) = \bigoplus_{w\in W} \mathbb{Z}\langle X_w \rangle .$$

'Same' problem as for $\mathcal{G}$: decompose products of classes into a linear combination of elementary classes. This is the *structure constants problem*.

Grassmanians as incomplete flag varieties

If P is the stabilizer of a k-vector subspace of $\mathbb{C}^n$, then one has

$$G(k,n) \simeq GL_n(\mathbb{C})/P.$$

Such a P is a *maximal parabolic subgroup*. It contains a conjugate of B. In general, a *parabolic subgroup* of G is a subgroup that contains a conjugate of the chosen Borel.

Definition

In the general setting, the variety G/P is projective ; and is called an *incomplete flag variety*.

The Grassmannian is thus an example of partial flag manifold, hence the name *Schubert calculus* for the geometry of G/B and, more generally, G/P...

Thank you