Homotopy, invariants and Serre fibrations

GdT Simplicial random variables

Arthur Garnier

Laboratoire Amiénois de Mathématique Fondamentale et Appliquée
Université de Picardie Jules Verne

March 17, 2021

Menu

Motivations

Principle : Associate to topological spaces some **algebraic invariants** to identify them. Different kinds of invariants :

- Number : dimension(s), Euler characteristic...
- Structures : (co)homology groups, **homotopy groups**, cohomology algebras...
- Elements in these structures : characteristic classes, fundamental class...

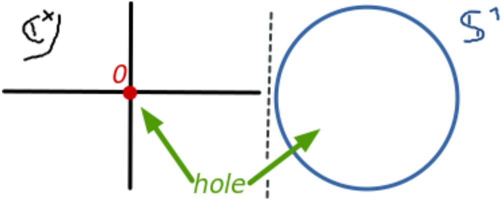
How to distinguish between spaces ? By looking at the relationships between them !

Another motivation: existence of geometric structures on spaces (Hopf-Poincaré, Brouwer and Lefschetz fixed point theorems...)

Motivations

Suitable notion of "morphism" for topology : continuous maps. *In the sequel, every map is continuous*.
Associated "Isomorphisms": **homeomorphisms**. Too strong !
Spaces may look the same, without being homeomorphic.

Example : $\mathbb{S}^1$ and $\mathbb{C}^\times$ are not homeomorphic...
...but both only have one "hole".



Two such spaces are **homotopy equivalent**.

Fundamental group

- What is a "hole", topologically?

$\simeq$

- Question : how to continuously deform a path into another ?

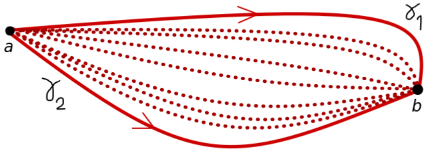
Definition

Let $I := [0,1]$ and X be a topological space. A **homotopy** between two paths with the same endpoints $\gamma_1, \gamma_2 : I \to X$ is a (continuous) map $H : I \times I \to X$ st

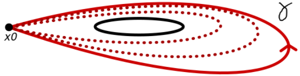
$$\forall t \in I, \begin{cases} H(t,0) = \gamma_1(t), \\ H(t,1) = \gamma_2(t), \end{cases} \text{and } \forall s \in I, \begin{cases} H(0,s) = \gamma_1(0) = \gamma_2(0), \\ H(1,s) = \gamma_1(1) = \gamma_2(1). \end{cases}$$

In this case, we denote $\gamma_1 \sim \gamma_2$.

Fundamental group



Case of loops :



We fix $x_0 \in X$. If $\gamma : I \to X$ is a loop based at $x_0 \in X$ (i.e. $x_0 = \gamma(0) = \gamma(1)$), we let

$$[\gamma] := \{\gamma' : I \to X \ ; \ \gamma'(0) = \gamma'(1) = x_0 \text{ and } \gamma' \sim \gamma\}$$

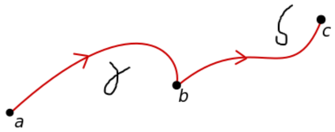
and

$$\pi_1(X, x_0) := \{[\gamma], \ \gamma \text{ loop in } X, \text{ based at } x_0\}.$$

Fundamental group

We may compose paths : If $\gamma_1 : a \to b$ and $\gamma_2 : b \to c$, then we let

$$\gamma * \delta : t \mapsto \begin{cases} \gamma(2t) & \text{si} \quad 0 \leq t \leq \frac{1}{2} \\ \delta(2t-1) & \text{si} \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$



$$\left.\begin{array}{l} \gamma_1 \sim \gamma_2 \\ \delta_1 \sim \delta_2 \end{array}\right\} \Rightarrow \gamma_1 * \delta_1 \sim \gamma_2 * \delta_2. \qquad (\star)$$

Using $(\star)$ we define

$$\forall [\gamma], [\delta] \in \pi_1(X, x_0), \; [\gamma] * [\delta] := [\gamma * \delta].$$

Key fact 1: $(\pi_1(X, x_0), *)$ is a group.

Fundamental group

If X is path-connected, then $\pi_1(X, x_0) \simeq \pi_1(X, y_0)$, $\forall x_0, y_0 \in X$.

Key fact 2: $\pi_1 : \mathfrak{Top}^* \to \mathfrak{Grp}$ is a functor.

Decrypted : for $f : (X, x_0) \to (Y, y_0)$ map of pointed spaces, then we have a group homomorphism

$$\pi_1(f) = f_1 \quad : \quad \begin{array}{ccc} \pi_1(X, x_0) & \to & \pi_1(Y, y_0) \\ [\gamma] & \mapsto & [f \circ \gamma] \end{array} \quad \text{st} \quad \begin{cases} (g \circ f)_1 = g_1 \circ f_1 \\ (id_X)_1 = id_{\pi_1(X)} \end{cases}$$

Example

Up to homotopy, a loop in $\mathbb{S}^1$ is determined by the number of its (oriented) turns around 0, hence $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$.

Fundamental group

We may generalize the notion of homotopy to maps between spaces :

Definition

- A **homotopy** between maps $f, g : X \to Y$ is a map $H : X \times I \to Y$ such that

$$\forall x \in X, \begin{cases} H(x,0) = f(x), \\ H(x,1) = g(x). \end{cases}$$

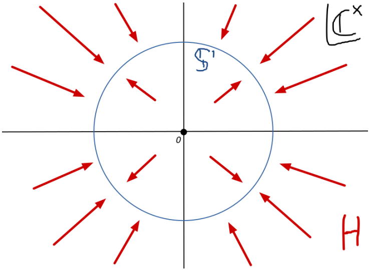
  In this case, we denote $f \sim g$.
- Two spaces X, Y are **homotopy equivalent** if there are $f : X \to Y$ and $g : Y \to X$ st $g \circ f \sim id_X$ and $f \circ g \sim id_Y$.

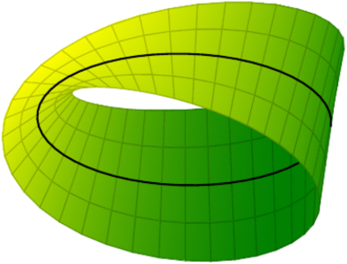
Fundamental group

Example

- If $\emptyset \neq X \subset \mathbb{R}^n$ is convex and $x_0 \in X$, then
$$H(x,t) = (1-t)x_0 + tx$$
is a homotopy $id_X \sim x_0$ (we say that X is *contractible*).
- Denote
$$\begin{array}{cccc} \iota & : & \mathbb{S}^1 & \hookrightarrow & \mathbb{C}^\times \\ & & z & \mapsto & z \end{array} \quad \text{and} \quad \begin{array}{cccc} p & : & \mathbb{C}^\times & \twoheadrightarrow & \mathbb{S}^1 \\ & & z & \mapsto & z/|z| \end{array}$$
We have $p \circ \iota = id_{\mathbb{S}^1}$ and a homotopy $\iota \circ p \sim id_{\mathbb{C}^\times}$ defined by
$$H(z,t) = tz + (1-t)z/|z|.$$

Fundamental group





Example

Another example: the Möbius strip retracts onto $\mathbb{S}^1$.

Fundamental group

Example

We have seen that $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$ and we have an isomorphism

$$\begin{array}{ccc} \pi_1(\mathbb{C}^\times) & \longrightarrow & \mathbb{Z} \\ [\gamma] & \longmapsto & \frac{1}{2i\pi}\oint_\gamma \frac{dz}{z} = \mathrm{Ind}_\gamma(0) \end{array}$$

Key fact 3: π_1 is homotopy invariant : if $X \sim Y$ then $\pi_1(X) \simeq \pi_1(Y)$.

Not a complete invariant! $\pi_1(\mathbb{S}^2) = 1 = \pi_1(\mathrm{pt})$ but $\mathbb{S}^2 \nsim \mathrm{pt}$.

Refine by looking at higher dimensions?

Higher homotopy groups

For any space X and $x_0 \in X$, we have defined

$$\pi_1(X, x_0) = [(\mathbb{S}^1, 1), (X, x_0)] \overset{\text{def}}{=} \{\gamma : \mathbb{S}^1 \to X \ ; \ \gamma(1) = x_0\}/\sim .$$

We can do the same for each $n \geq 2$:

$$\pi_n(X, x_0) := [(\mathbb{S}^n, 1), (X, x_0)].$$

We have a homeomorphism of pairs $(I^n, \partial I^n) \simeq (\mathbb{S}^n, 1)$, hence $\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)]$ and we have a composition

$$\gamma * \delta : (t_1, \ldots, t_n) \mapsto \begin{cases} \gamma(2t_1, t_2, \ldots, t_n) & \text{si} \quad 0 \leq t_1 \leq \frac{1}{2}, \\ \delta(2t_1 - 1, t_2, \ldots, t_n) & \text{si} \quad \frac{1}{2} \leq t_1 \leq 1. \end{cases}$$

Higher homotopy groups

Compatible with homotopy and makes $\pi_n(X, x_0)$ an **abelian group**. For each $n \geq 2$, $\pi_n : \mathfrak{Top}^* \to \mathfrak{Ab}$ is a functor (i.e. $f : (X, x_0) \to (Y, y_0)$ induces $f_n : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ st $(f \circ g)_n = f_n \circ g_n$ and $id_n = id$) and a homotopy invariant (i.e. $f \sim g \;\Rightarrow\; f_n = g_n$).

Adding $\pi_0(X, x_0) := \{\text{path-connected components of } X\}$, we obtain algebraic homotopy invariants $\pi_n(X, x_0)$ for a pointed space (X, x_0) and $n \in \mathbb{N}$.

Does this determine the homotopy type? No!

Higher homotopy groups

Example

The map

$$f : \mathbb{N} \to \{0\} \cup \{1/n,\ n \in \mathbb{N}^*\}$$
$$n \mapsto \begin{cases} 1/n & \text{si } n \neq 0 \\ 0 & \text{sinon} \end{cases}$$

is not a homotopy equivalence but f_n is an isomorphism for every n.

Definition

A map $f : (X, x_0) \to (Y, y_0)$ is a **weak homotopy equivalence** if $f_n : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism for each $n \geq 1$ and one-to-one for $n = 0$.

Partial converse: CW-complexes and Whitehead's theorem

We have *homotopy equivalence* $\Rightarrow$ *weak homotopy equivalence*.
There is a class of spaces for which the converse holds.

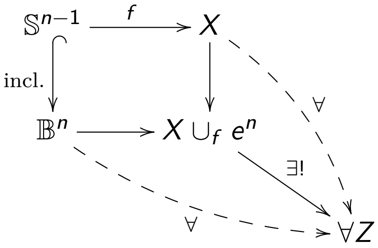
Definition

A **CW-complex** is a space X obtained from a set of points X_0 by inductively gluing n-cells.
More precisely, non-decreasing sequence of subspaces $X_0 \subset X_1 \subset \cdots \subset X$ st X_0 is discrete, for $n > 0$, X_n is obtained from X_{n-1} by attaching n-balls along their boundaries.

If infinitely many cells, weak topology on X: $A \subset X$ is closed iff $A \cap X_n$ is closed in X_n for all n. This is important in Ivan's work.

CW-complexes

Attaching n-cell ? Start with a space X and a map $f : \mathbb{S}^{n-1} \to X$.
Commutative diagram (pushout)

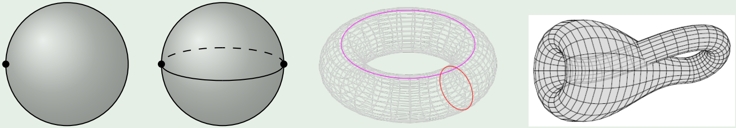


A CW-complex X is Hausdorff, each open cell is homeomorphic to an open ball and open cells form a partition of X.

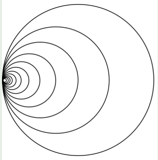
CW-complexes

Example

- Spheres, torus, Klein's bottle...



- But also whole classes of spaces: projective spaces, (realisations of) simplicial complexes, smooth manifolds and complex algebraic varieties (they are even "triangulated")...
- ...but some are not : the Hawaiian earring



Whitehead's theorem

- First advantage : "easily" compute invariants (Euler characteristic, (co)homology, π_1 by generators and relations, and (heavy) algorithms for $\pi_{n\geq 2}$'s).
- Second advantage : weak homotopy equivalences and homotopy equivalences coincide on CW-complexes :

Theorem (*J. H. C. Whitehead, 1949*)

If X, Y space with the homotopy type of CW-complexes and if $f : X \to Y$ is a weak equivalence, then it is a homotopy equivalence.

categorical generalization : fibrant-cofibrant objects in model categories...

Reminder on complexes and exact sequences

A (chain) **complex** $C_\bullet = (C_n, d_n)_{n\in\mathbb{Z}}$ is a sequence of abelian groups and homomorphisms of the form

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

such that $d_n \circ d_{n+1} = 0$. We introduce its **homology**

$$\forall n \in \mathbb{Z},\ \ H_n(C_\bullet) := \ker(d_n)/\mathrm{im}(d_{n+1}).$$

morphisms between complexes ? sequence $(\varphi_n : C_n \to D_n)$ st

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \\
\cdots & \longrightarrow & D_{n+1} & \longrightarrow & D_n & \longrightarrow & D_{n-1} & \longrightarrow & \cdots
\end{array}$$

We associate to $\varphi_\bullet$ homomorphisms $H_n(\varphi_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet)$.

Reminder on complexes and exact sequences

Homological algebra : general vocabulary for constructions of algebraic invariants of spaces.

Key point : if $\varphi : A_\bullet \to B_\bullet$ and $\psi : B_\bullet \to C_\bullet$ st

$$\forall n \in \mathbb{Z}, \quad 0 \longrightarrow A_n \xrightarrow{\varphi_n} B_n \xrightarrow{\psi_n} C_n \longrightarrow 0 \text{ is a SES}$$

i.e.

$$\mathrm{im}(\psi_n) = C_n, \ \ker(\psi_n) = \mathrm{im}(\varphi_n) \text{ and } \ker(\varphi_n) = 0,$$

then we get a LES

$$\cdots H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{H_n(\varphi)} H_n(B) \xrightarrow{H_n(\psi)} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \cdots$$

Crucial feature for applications in algebraic topology.

Homotopy analogue ?

Question : what is the suitable notion of SES in $\mathfrak{Top}$ to get a LES in homotopy? Serre fibration !

Definition

A map $p : E \to B$ is **Serre fibration** if we may complete every diagram of the form ($n \in \mathbb{N}$ and $I = [0,1]$)

$$
\begin{array}{ccccc}
I^n & \xrightarrow{\sim} & I^n \times \{0\} & \xrightarrow{\forall f_0} & E \\
 & & \iota \big\downarrow & \overset{\exists \widetilde{f}}{\nearrow} & \big\downarrow p \\
I^{n+1} & \xrightarrow{\sim} & I^n \times I & \xrightarrow[\forall f]{} & B
\end{array}
$$

In this case, we say that p has the **homotopy lifting property** (HLP) with respect to the pair $(I^{n+1}, I^n \times \{0\})$.

Variant and some properties

If a map $p : E \to B$ has the HLP for every pair $(Z \times I, Z)$ (any Z), then it is called a **Hurewicz fibration**. But this is too strong!

Proposition

- *We have*

$$\mathrm{HLP}//(I^n, I^{n-1} \times \{0\}) \Leftrightarrow \mathrm{HLP}//(\mathbb{B}^n, \mathbb{S}^{n-1})$$

$$\Leftrightarrow \mathrm{HLP}//(I^n, J^{n-1}),$$

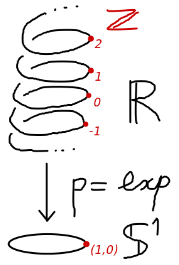
  where $J^n := (I^n \times \{0\}) \cup_{\partial I^n \times \{0\}} (\partial I^n \times I)$*. For instance,* J^2 *is a lidless box.*
  $n = 0 \Rightarrow p(E)$ *is union of 0-connected components of* B*.*
- *Every fiber bundle (hence any covering map) is a Serre fibration.*

Examples of Serre fibrations

Example

- The exponential map is a $\mathbb{Z}$-covering map

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\exp} & \mathbb{S}^1 \\ x & \longmapsto & e^{2i\pi x} \end{array}$$

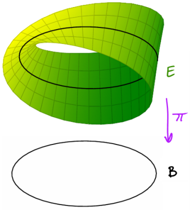


- Quaternions give universal covering $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{S}^3 \twoheadrightarrow SO(3)$.

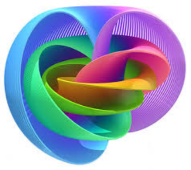
Examples of Serre fibrations

Example

- The Möbius strip $[-1,1] \hookrightarrow \mathcal{M} \overset{\pi}{\twoheadrightarrow} \mathbb{S}^1$



- The Hopf bundle $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \twoheadrightarrow \mathbb{S}^2$



Homotopy long exact sequence

First remark: we have bijections (for a space X and $x_0 \in X$)

$$\forall n \geq 1,\ \pi_n(X, x_0) \overset{\text{def}}{=} [(\mathbb{S}^n, 1), (X, x_0)]$$

$$\simeq [(I^n, \partial I^n), (X, x_0)] \approx [(\mathbb{B}^n, \mathbb{S}^{n-1}), (X, x_0)].$$

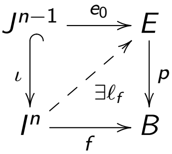
Fix $p : E \to B$ Serre fibration (surjective if B is 0-connected), $b_0 \in B$, $F := p^{-1}(b_0) \overset{\iota}{\hookrightarrow} E$ and $e_0 \in F$. We have induced maps $\iota_n : \pi_n(F, e_0) \to \pi_n(E, e_0)$ and $p_n : \pi_n(E, e_0) \to \pi_n(B, b_0)$ and we want to construct a **connecting homomorphism**

$$\partial : \pi_n(B, b_0) \to \pi_{n-1}(F, e_0),$$

fitting in a homotopy LES involving ι_n and p_n (as for homology).

Idea of construction of the homotopy LES

Start with $[f] \in \pi_n(B, b_0)$, represented by $f : (I^n, \partial I^n) \to (B, b_0)$.
We have a diagram



restricts to $\ell_f : I^{n-1} \times \{1\} \to F$ and $\widetilde{f} := (I^{n-1}, \partial I^{n-1}) \xrightarrow{\ell_f} (F, e_0)$.
Define

$$\partial([f]) := [\widetilde{f}].$$

By HLP again, this is well-defined and gives a morphism

$$\pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0).$$

Theorem and consequences

Theorem

If $p : E \to B$ Serre fibration, $e_0 \in E$, $b_0 := p(e_0)$ and $F = p^{-1}(b_0)$, then there are connecting homomorphisms $\partial : \pi_n(B, b_0) \to \pi_{n-1}(F, e_0)$ fitting into a long exact sequence

$$\cdots \xrightarrow{p_{n+1}} \pi_{n+1}(B, b_0) \xrightarrow{\partial} \pi_n(F, e_0) \xrightarrow{\iota_n} \pi_n(E, e_0) \xrightarrow{p_n} \pi_n(B, b_0) \xrightarrow{\partial} \cdots$$

$$\cdots \xrightarrow{p_1} \pi_1(B, b_0) \xrightarrow{\partial} \pi_0(F, e_0) \xrightarrow{\iota_0} \pi_0(E, e_0) \xrightarrow{p_0} \pi_0(B, b_0).$$

Proof heavily uses the HLP (see [Félix & Tanré], Théorème 7.14 or [Hatcher], Theorem 4.41).

Theorem and consequences

Remark

Doesn't depend on the fiber: we have $p^{-1}(x) \sim p^{-1}(y)$ *for* $x, y \in B$.

Corollary

Is $p : E \to B$ is a covering map, then p_n is an isomorphism for every $n \geq 2$. Moreover, if E is 0-connected, then the sequence reduces to

$$1 \longrightarrow \pi_1(E) \xrightarrow{p_1} \pi_1(B) \xrightarrow{\partial} \pi_0(F) \longrightarrow 0$$

Examples of applications

Example

- From $\exp : \mathbb{R} \twoheadrightarrow \mathbb{S}^1$, we get $\pi_{i\geq 2}(\mathbb{S}^1) = 0$ and the sequence

$$1 \longrightarrow \pi_1(\mathbb{R}) = 1 \longrightarrow \pi_1(\mathbb{S}^1) \xrightarrow[\sim]{\partial} \pi_0(\mathbb{Z}) = \mathbb{Z} \longrightarrow 0.$$

- The universal covering $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{S}^3 \twoheadrightarrow SO(3)$ yields

$$\pi_1(SO(3)) \simeq \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \pi_i(SO(3)) = \pi_i(\mathbb{S}^2),\ \forall i \geq 2.$$

- The antipode $p : \mathbb{S}^n \twoheadrightarrow \mathbb{P}^n(\mathbb{R})$ gives isomorphisms

$$\pi_i(\mathbb{P}^n(\mathbb{R})) \simeq \pi_i(\mathbb{S}^n),\ \forall i \geq 2.$$

Examples of applications

The $\pi_i(\mathbb{S}^n)$'s are a mess: for instance, $\pi_3(\mathbb{S}^2) = \mathbb{Z}$ and $\pi_{n+1}(\mathbb{S}^n) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$. Still there are algorithms for it, e.g. the Kenzo algorithm of Sergeraert.

Corollary

If $p : E \to B$ Serre fibration with contractible fibers (i.e. $p^{-1}(x) \sim \mathrm{pt}$), then p is a weak homotopy equivalence.
If in addition E and B both have the homotopy type of a CW-complex, then p is a homotopy equivalence.

To be used in the talks to come...

Concluding remarks

Remark

- *About homology, we have Hurewicz morphisms (for X 0-connected)*

$$\pi_n(X) \to H_n(X, \mathbb{Z})$$

and an isomorphism

$$\pi_1(X)^{\mathrm{ab}} \to H_1(X, \mathbb{Z}).$$

- *If* $f : X \to Y$ *map between 1-connected CW-complexes, then* f *homotopy equivalence iff* $H_*(f) : H_*(X, \mathbb{Z}) \to H_*(Y, \mathbb{Z})$ *iso.*
- *Sadly, no LES for homology, but a "spectral sequence"...*

Thank you !

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2\times\mathbb{Z}/12$	$(\mathbb{Z}/2)^2\times\mathbb{Z}/84$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2\times\mathbb{Z}/12$	$(\mathbb{Z}/2)^2\times\mathbb{Z}/84$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}\times\mathbb{Z}/12$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/3\times\mathbb{Z}/24$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2\times\mathbb{Z}/12\times\mathbb{Z}/120$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/30$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/60$	$\mathbb{Z}/2\times\mathbb{Z}/24$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/120$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0
S^{10}	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0
S^{11}	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
S^{12}	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^{13}	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
S^{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$