THE CLASSICAL THEORY OF SOERGEL BIMODULES AND THE LINK WITH LUSZTIG'S CONJECTURE

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Abstract. These notes are taken from a talk I gave at the Arbeitsgemeinschaft on Geometric Representation Theory that took place at Oberwolfach on April 2022.

The aim is to give an introduction to Soergel bimodules. After some geometric motivations, we define Bott-Samelson and Soergel bimodules. We review their main properties (filtrations, character) and in particular, state the Soergel categorification theorem.

Then, we discuss the relation between Soergel's conjecture and Lusztig's multiplicity conjecture, via the Soergel modules.

Contents

Part 1. Geometric introduction

Let G be a connected reductive algebraic group over $\mathbb{C}$ and choose a Borel $B < G$ and a maximal torus $T < B$. If $X^*(T)$ denotes the character lattice of T, then we consider the symmetric algebra $R := \mathrm{Sym}(X^*(T) \otimes \mathbb{Q}) = \mathrm{Sym}(\mathfrak{h}^*) \simeq H^*_T(\mathrm{pt}, \mathbb{Q})$, a graded algebra with $\deg(\mathfrak{h}^*) = 2$, $\mathfrak{h}$ being the Lie algebra of T. We have an isomorphism of R-algebras

$$H^*_T(G/B, \mathbb{Q}) = H^*_B(G/B, \mathbb{Q}) \simeq R \otimes_{R^W} R,$$

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where $W = N_G(T)/T$ is the (finite) Weyl group. We consider the B-equivariant derived category $D^b_B(G/B,\mathbb{Q})$ and the hypercohomology

$$\mathbb{H}^*_B : D^b_B(G/B,\mathbb{Q}) \to H^*_T(G/B,\mathbb{Q})-\mathbf{gmod} = (R \otimes_{R^W} R)-\mathbf{gmod},$$

where **gmod** stands for graded modules, yields a functor

$$D^b_B(G/B,\mathbb{Q}) \xrightarrow{\mathbb{H}^*_B} (R \otimes_{R^W} R)-\mathbf{gmod} \xrightarrow{\mu^*} (R \otimes R)-\mathbf{gmod} = R-\mathbf{gbim},$$

where μ is the canonical map $R\otimes R \to R \otimes_{R^W} R$ and **gbim** is the category of R-bimodules. We still abusively denote this functor by $\mathbb{H}^*_B$.

On the other hand, we have the *geometric Hecke category*

$$\mathcal{H} = \mathcal{H}_{\mathrm{geom}} := \langle IC_w,\ w \in W\rangle_{\oplus,[1]} \simeq \left\langle \underline{\mathbb{Q}}_{P_s/B},\ s \in S\right\rangle_{*,\oplus,[1],\mathrm{Kar}},$$

where $IC_w := IC(\overline{BwB/B}) \in D^b_B(G/B,\mathbb{Q})$ is the intersection complex of the Schubert variety associated to $w \in W$. We have also denoted by $*$ the convolution product of B-equivariant sheaves, [1] is the homological shift and Kar is the Karoubian completion (i.e. completion under direct summands). Moreover, $P_s := \overline{BsB}$ is the minimal parabolic subgroup generated by the simple reflection $s \in S$. The split Grothendieck group $[\mathcal{H}]_\oplus$ is a ring under the product $[\mathcal{F}]\cdot[\mathcal{G}] := [\mathcal{F} * \mathcal{G}]$, and we endow it with the structure of a $\mathbb{Z}[v,v^{-1}]$-algebra via the rule $v\cdot[\mathcal{F}] := [\mathcal{F}[1]]$. The key observation of Soergel is the following theorem:

Theorem 1.1. *The restricted functor*

$$\mathbb{H}^*_B : \mathcal{H} \longrightarrow R-\mathbf{gbim}$$

is monoidal and fully faithful, so it is an equivalence onto its essential image. This image is the category SBim *of Soergel bimodules.*

Let $w \in W$ and consider an expression $\underline{w} = (s,t,\dots,u)$ for w, with $s,t,\dots,u \in S$. We define the *Bott-Samelson space*

$$\mathrm{BS}_{\underline{w}} := P_s \times_B P_t \times_B \cdots \times_B P_u/B$$

and the multiplication map $\xi : \mathrm{BS}_{\underline{w}} \to \overline{BwB/B}$, which is a resolution of singularities of the Schubert variety and we let

$$\mathcal{E}_{\underline{w}} := \xi_*\underline{\mathbb{Q}}_{\mathrm{BS}_{\underline{w}}} = \underline{\mathbb{Q}}_{P_s/B} * \underline{\mathbb{Q}}_{P_t/B} * \cdots * \underline{\mathbb{Q}}_{P_u/B}[\ell(\underline{w})] \in \mathcal{H}.$$

We call $\mathbb{H}^*_B(\mathcal{E}_{\underline{w}}) =: \mathrm{BS}(\underline{w}) \in R-\mathbf{gbim}$ is the *Bott-Samelson bimodule* associated to the expression $\underline{w}$. If $s \in S$, then

$$\mathrm{BS}(\underline{s}) = \mathbb{H}^*_B(\underline{\mathbb{Q}}_{P_s/B}[1]) = \mathbb{H}^*_B(IC_s) \simeq R \otimes_{R^s} R(1),$$

where (1) is the grading shift.

The central result of the theory of Soergel bimodules is the following categorification result, that we shall detail and state properly below:

Theorem 1.2. *The map*

$$\begin{array}{ccc} H & \longrightarrow & [\mathcal{H}]_\oplus \\ \underline{H}_s & \longmapsto & [\underline{\mathbb{Q}}_{P_s/B}[1]] \end{array}$$

is an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras, where H is the Hecke algebra of (W,S) and $\underline{H}_s$ is the element of the Kazhdan-Lusztig basis associated to $s \in S$. Moreover, its reciprocal $[\mathcal{H}]_\oplus \xrightarrow{\sim} [\mathrm{SBim}]_\oplus \xrightarrow{\sim} H$ uses the "character" of a Soergel bimodule.

The combinatorial description of Soergel bimodules still makes sense for any Coxeter group, leading to a categorification of the Hecke algebra.

Part 2. Definitions and first properties of Bott-Samelson and Soergel bimodules

1. Setting the stage: Demazure operators, bimodules and tensor product

Let (W,S) be a Coxeter system and let $W \to GL(\mathfrak{h})$ be the geometric representation of W over $\mathbb{R}$. We denote by $(\alpha_s)_{s\in S}$ the standard basis of $\mathfrak{h}$ and we let $R := \mathrm{Sym}(\mathfrak{h}) = \mathbb{R}[\alpha_s,\ s \in S]$, with $\deg(\alpha_s) = 2$ for all $s \in S$.

If M is a graded left R-module and $p = \sum_i p_i v^i \in \mathbb{N}[v^{\pm 1}]$, we let $M^{\oplus p} := \sum_{i\in\mathbb{Z}} M(i)^{\oplus p_i}$, where $M(1)$ is the grading shift. In other words, it extends the rule $v \cdot M := M(1)$ and yields a $\mathbb{Z}[v^{\pm 1}]$-action on $R-\mathbf{gmod}$. If $M \simeq R^{\oplus p}$ for some $p \in \mathbb{N}[v^{\pm 1}]$, the we say that M is *graded free* of graded rank $\underline{\mathrm{rk}}(M) =: p$.

We now define the Demazure operators:

Definition 2.1. *The* Demazure operator ∂_s *associated to* $s \in S$ *is the graded map*

$$\begin{array}{cccc} \partial_s & : & R & \longrightarrow & R^s(-2) \\ & & f & \longmapsto & \frac{f - s(f)}{\alpha_s} \end{array}$$

This is the projection associated to the decomposition $R \simeq R^s \oplus R^s\alpha_s = R^s \oplus R^s(-2)$, *as (left)* R^s*-modules.*

Proposition 2.2. *The Demazure operators satisfy the following properties*

(1) *For* $s \in S$, ∂_s *is a map of* R^s*-bimodules.*
(2) *For* $s \in S$, *we have* $s \circ \partial_s = \partial_s$, $\partial_s \circ s = -\partial_s$, $\partial_s^2 = 0$.
(3) *For* $s \in S$ *and* $f, g \in R$, *we have* $\partial_s(fg) = \partial_s(f)g + s(f)\partial_s(g)$.
(4) *The* R^s*-pairing*

$$\begin{array}{ccc} R \otimes R & \longrightarrow & R^s \\ f \otimes g & \longmapsto & \partial_s(fg) \end{array}$$

is perfect.
(5) *If* $s, t \in S$ *are such that* $m_{s,t} < \infty$, *then we have the braid relation*

$$\underbrace{\partial_s\partial_t\cdots}_{m_{s,t}} = \underbrace{\partial_t\partial_s\cdots}_{m_{s,t}}.$$

We also let $R-\mathbf{gbim}$ be the full subcategory of R-bimodules, which are finitely generated as left and as right R-modules. It is a monoidal subcategory. For simplicity, if M, N are R-bimodules, we denote $MN := M \otimes_R N$.

2. Bott-Samelson and Soergel bimodules

Definition 2.3. (1) *For* $s \in S$, *let*

$$B_s := R \otimes_{R^s} R(1) \in R-\mathbf{gbim}.$$

(2) *If* $\underline{w} = (s, t, \ldots, u)$ *is an expression for* $w \in W$, *then we let*

$$\mathrm{BS}(\underline{w}) := B_sB_t\cdots B_u \simeq R \otimes_{R^s} R \otimes_{R^t} \cdots \otimes_{R^u} R(\ell(\underline{w})) \in R-\mathbf{gbim}.$$

(3) *The* category of Bott-Samelson bimodules BSBim *has Bott-Samelson bimodules as objects and, for* $B, B' \in \mathrm{BSBim}$,

$$\mathrm{Hom}_{\mathrm{BSBim}}(B,B') := \mathrm{Hom}^{\bullet}_R(B,B') = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{R-\mathbf{gbim}}(B, B'(i)).$$

(4) *A* Soergel bimodule *is a direct summand of a finite direct sum of (grading) shifts of Bott-Samelson bimodules. The* category of Soergel bimodules SBim *is the (strictly) full subcategory of* $R-\mathbf{gbim}$ *with Soergel bimodules as objects. In other words,*

$$\mathrm{SBim} = \langle Ob(\mathrm{BSBim})\rangle_{\oplus,(1),\mathrm{Kar}} = \langle R, B_s,\ s \in S\rangle_{\oplus,\otimes,(1),\mathrm{Kar}}.$$

Remark 2.4. *For any two expressions* $\underline{u}$ *and* $\underline{v}$, *we have* $\mathrm{BS}(\underline{u})BS(\underline{v}) = \mathrm{BS}(\underline{uv})$, *so that the category* BSBim *is monoidal.*

Proposition 2.5. *The category* SBim *is monoidal and Krull-Schmidt (i.e. any object has a unique decomposition into objects having local endomorphism rings).*

Example 2.6. *(1) If $s \in S$, then R and B_s are indecomposable bimodules and we have*

$$\begin{aligned} B_s B_s = R \otimes_{R^s} R \otimes_{R^s} R(2) &\simeq R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) \\ &= R \otimes_{R^s} R(2) \oplus R \otimes_{R^s} R = B_s(1) \oplus B_s(-1). \end{aligned}$$

Thus, the set of isomorphism classes of indecomposable Soergel bimodules in type A_1, up to shift, is given by $\{R, B_s\}$. Notice also that, in the Hecke algebra $H(W,S)$, we have $\underline{H}_s^2 = v\underline{H}_s + v^{-1}\underline{H}_s$, which is a "decategorified" version of the isomorphism $B_s B_s \simeq B_s(1) \oplus B_s(-1)$ we just observed.

(2) If $s,t \in S$ are such that $2 < m_{s,t} < \infty$, then $B_s B_t \simeq R \otimes_{R^s} R \otimes_{R_t} R(2)$ and $B_t B_s$ are generated by $1 \otimes 1 \otimes 1$, so they are indecomposable. Let $B_{st} := B_s B_t$ and $B_{ts} := B_t B_s$ and, since $B_s B_s$ is decomposable, we see that the bimodules B_s and B_t are not isomorphic.

(3) Suppose that $W = \mathfrak{S}_3$ is of type A_2, so that $m_{s,t} = 3$. We already know that $R, B_s, B_t, B_{st}, B_{ts}$ are indecomposable. Let $B_{sts} := R \otimes_{R^{s,t}} R(3)$. It is generated by $1 \otimes 1$ in degree -3, so it is indecomposable. The maps

$$\begin{array}{ccc} B_{sts} & \xrightarrow{\iota_{sts}} & B_s B_t B_s \\ 1 \otimes 1 & \longmapsto & 1 \otimes 1 \otimes 1 \otimes 1 \end{array} \quad \text{and} \quad \begin{array}{ccc} B_s & \xrightarrow{\iota_s} & B_s B_t B_s \\ 1 \otimes 1 & \longmapsto & \frac{1}{2}(1 \otimes \alpha_t \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \alpha_t \otimes 1) \end{array}$$

are injective. We also have a projection

$$\begin{array}{ccc} B_s B_t B_s & \xrightarrow{p_s} & B_s \\ 1 \otimes f \otimes g \otimes 1 & \longmapsto & -\partial_s(fg) \otimes 1 \end{array}$$

satisfying $p_s \circ \iota_s = \mathrm{id}_{B_s}$ and $p_s \circ \iota_{sts} = 0$. The endomorphism $e_s := \iota_s \circ p_s$ of $B_s B_t B_s$ is idempotent and so is $e_{sts} := 1 - e_s$. We obtain a decomposition

$$B_s B_t B_s \simeq B_{sts} \oplus B_s.$$

Similarly, we have $B_t B_s B_t \simeq B_{sts} \oplus B_t$. Observe finally that

$$B_s B_t B_s \simeq B_{sts} \oplus B_s \not\simeq B_{st}(-1) \oplus B_{st}(1) \simeq B_s B_s B_t,$$

so that $B_{st} \not\simeq B_{ts}$ and also,

$$B_{sts} B_s \simeq B_s B_{sts} \simeq B_{sts}(1) \oplus B_{sts}(-1) \simeq B_{sts} B_t \simeq B_t B_{sts}.$$

Therefore, the set $\{R, B_s, B_t, B_{st}, B_{ts}, B_{sts}\}$ is a complete set of distinct indecomposable Soergel bimodules, up to shift and isomorphism. This set is parametrized by $W = \mathfrak{S}_3$, which is a general feature.

Part 3. The classical theory: standard bimodules, filtrations and characters

3. Twisted bimodules and standard filtrations

Definition 3.1. *For $x \in W$, consider the R-module automorphism $\mu_x : R \to R$ defined by $\mu_x(r) := xr$ and consider the* twisted bimodule *$R_x := R_{\mu_x} \in R\text{–}\mathbf{gbim}$ (i.e. $R \otimes R$ acts on R_x via $((r_1, r_2) \cdot r := r_1 r x r_2)$. The* category of standard bimodules *is the full subcategory*

$$\mathrm{StdBim} := \langle R_x,\ x \in W \rangle_{\oplus,(1)} \subset R\text{–}\mathbf{gbim}.$$

Remark 3.2. *For $x, y \in W$, we have $R_x \otimes R_y \simeq R_{xy}$ and $\mathrm{Hom}^\bullet(R_x, R_y) = \delta_{x,y} R$ (as graded vector spaces). Moreover, the category* StdBim *is closed under direct summands.*

Proposition 3.3. *We have an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras*

$$\begin{array}{ccc} \mathbb{Z}[v^{\pm 1}][W] & \xrightarrow{\sim} & [\mathrm{StdBim}]_\oplus \\ x & \longmapsto & [R_x] \end{array}$$

In other words, the category of standard bimodules categorifies the group algebra of W over $\mathbb{Z}[v^{\pm 1}]$. Since SBim *categorifies the Hecke algebra, we can say that* SBim *is a "deformation" of* StdBim*.*

4. Standard filtrations and characters of Soergel bimodules

For $s \in S$, we have $B_s = R \otimes_{R^s} R(1)$ and consider the elements $c_1, c_s, d_s \in B_s$ defined by

$$c_1 := 1 \otimes 1,\ c_s := \tfrac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \quad \text{and} \quad d_s := \tfrac{1}{2}(\alpha_s \otimes 1 - 1 \otimes \alpha_s).$$

For $f \in R$, we have the following relations:

$$f \cdot c_1 = c_1 \cdot f + d_s \cdot \partial_s(f) = c_1 \cdot s(f) + c_s \cdot \partial_s(f) \quad \text{and} \quad f \cdot d_s = d_s \cdot s(f).$$

These lead to (non-split) short exact sequences

$$A \longrightarrow B$$

$$(\Delta_s) \qquad 0 \longrightarrow R_s(-1) \xrightarrow{1 \mapsto d_s} B_s \xrightarrow{f \otimes g \mapsto fg} R(1) \longrightarrow 0$$

and

$$(\nabla_s) \qquad 0 \longrightarrow R(-1) \xrightarrow{1 \mapsto c_s} B_s \xrightarrow{f \otimes g \mapsto f \cdot s(g)} R_s(1) \longrightarrow 0.$$

Remark 3.4. *We have $R_s \subset B_s$ but $R_s \notin$* SBim*, so that* SBim *is not closed under submodules and quotients. In particular, this category is not abelian.*

For an expression $\underline{w} = (s, t, \ldots, u)$, tensoring the sequences $(\Delta_s), (\Delta_t), \ldots, (\Delta_u)$ with (shifts of) R, R_s, B_s... leads to a filtration of $BS(\underline{w})$. A subquotient of this filtration is a tensor product of $R(1)$ and $R_{s_i}(-1)$ and the choice of a subquotient corresponds to the choice of a subexpression of $\underline{w}$. Of course, we can do the same with (∇_s) instead of (Δ_s).

Fix an enumeration $x_0, x_1, \ldots,$ of W such that, if $x_i \leq x_j$ in the Bruhat order, then $i \leq j$. For instance, we can sort the elements by lengths, and put an arbitrary order on elements of the same length.

Definition 3.5. (1) *For such an enumeration, a* Δ-filtration *of a Soergel bimodule B is a filtration $0 = B^k \subset B^{k-1} \subset \ldots \subset B^0 = B$ with subquotients*

$$B^i / B^{i+1} \simeq R_{x_i}^{\oplus h_{x_i}},\ \text{with } h_{x_i} \in \mathbb{N}[v^{\pm 1}].$$

(2) *Dually, a* ∇-filtration *is a filtration $0 = B^0 \subset B^1 \subset \ldots \subset B^k = B$ such that $B^{i+1}/B^i \simeq R_{x_i}^{\oplus h'_{x_i}}$.*

The main interest of such filtrations relies in the following result:

Theorem 3.6 (Soergel)**.** *Any Soergel bimodule has a unique Δ-filtration (and a unique ∇-filtration). Moreover, for $x \in W$, the graded multiplicity h_x of R_x in the Δ-filtration (resp. the multiplicity h'_x of R_x in the ∇-filtration) depends only on B and x (not on the chosen enumeration). We can then define the multiplicity $h_x(B)$ (resp. $h'_x(B)$) of R_x in any Δ-filtration (resp. ∇-filtration) of B.*

Definition 3.7. *The* Δ-character *of a Soergel bimodule B is defined by*

$$\mathrm{ch}(B) = \mathrm{ch}_\Delta(B) := \sum_{x \in W} v^{\ell(x)} h_x(B) H_x \in H(W, S),$$

where $(H_x)_x$ is the standard basis of the Hecke algebra $H(W, S)$. Dually, we can define the ∇-character $\mathrm{ch}_\nabla(B) \in H(W,S)$ *of B.*

Example 3.8. *For $s \in S$, we have the Δ-filtration $0 \subset R_s(-1) \subset B_s$, with subquotients $R_s^{\oplus v^{-1}}$ and $R^{\oplus v}$, so that $\mathrm{ch}(B_s) = H_s + v = \underline{H}_s$.*

For $B, B' \in \mathrm{SBim}$, we have

$$\mathrm{ch}(B \oplus B') = \mathrm{ch}(B) + \mathrm{ch}(B') \quad \text{and} \quad \mathrm{ch}(B(\pm 1)) = v^{\pm 1}\mathrm{ch}(B).$$

Therefore, the character induced a $\mathbb{Z}[v^{\pm 1}]$-linear map

$$\mathrm{ch} : [\mathrm{SBim}]_{\oplus} \to H(W, S).$$

5. Localization

Let $Q := \mathrm{Frac}(R)$ be the fraction field of R. We have $B_s \otimes_R Q \simeq Q \otimes_{Q^s} Q$ as (R, Q)-bimodules.

Lemma 3.9. *(1) For an expression $\underline{x} = (s, t, \ldots, u)$, we have an isomorphism of (R, Q)-bimodules*

$$\mathrm{BS}(\underline{x}) \otimes_R Q \simeq Q \otimes_{Q^s} Q \otimes_{Q^t} \cdots \otimes_{Q^u} Q.$$

(2) There is an isomorphism of Q-bimodules

$$B_s \otimes_R Q \simeq Q_s \oplus Q.$$

Remark 3.10. *The short exact sequences $(\Delta_s) \otimes Q$ and $(\nabla_s) \otimes Q$ split for any $s \in S$. Hence, any $\Delta(\nabla)$-filtration splits over Q.*

Definition 3.11. *(1) The category BSBim_Q of Bott-Samelson bimodules over Q is the full monoidal subcategory of $Q-$**bim** generated by the bimodules BsQ for $s \in S$.*

(2) The category SBim_Q of Soergel bimodules over Q is the closure of BSBim under finite direct sums and direct summands.

*(3) The category StdBim_Q of standard bimodules over Q is the full subcategory of $Q-$**bim** consisting of direct summands of finite direct sums of twisted bimodules Q_x for $x \in W$.*

The monoidal functor $- \otimes_R Q : \mathrm{BSBim} \to \mathrm{BSBim}_Q$ induces a *localization functor*

$$\mathrm{Loc} : \mathrm{SBim} \longrightarrow \mathrm{SBim}_Q$$

and we have an equivalence of categories $\mathrm{Sbim}_Q \simeq \mathrm{StdBim}_Q$. In particular, we have an isomorphism of rings

$$\begin{array}{ccc} [\mathrm{SBim}_Q]_{\oplus} & \xrightarrow{\sim} & \mathbb{Z}[W] \\ {[Q_x]} & \longmapsto & x \end{array}$$

Therefore, we can see the localization as a "categorification after specializing at $v = 1$". More precisely, the following square commutes

$$\begin{tikzcd} \mathrm{SBim} \arrow[r, squiggly, "\mathrm{ch}"] \arrow[d, "\mathrm{Loc}"] & H(W,S) \arrow[d, "v \mapsto 1"] \\ \mathrm{SBim}_Q \arrow[r, squiggly, "\mathrm{ch}"'] & \mathbb{Z}[W]. \end{tikzcd}$$

6. Soergel's categorification theorem and Soergel's conjecture

Theorem 3.12 (Soergel)**.** *(1) There is an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras*

$$\begin{array}{cccc} c : & H(W, S) & \longrightarrow & [\mathrm{SBim}]_{\oplus} \\ & \underline{H}_s & \longmapsto & [B_s] \end{array}$$

(2) There is a bijection

$$\begin{array}{ccc} W & \xleftrightarrow{1:1} & \{\textit{indecomposables of } \mathrm{SBim}\} / \simeq, (1) \\ w & \longmapsto & B_w \end{array}$$

where B_w is a direct summand of $\mathrm{BS}(\underline{w})$ for any reduced expression $\underline{w}$ for $w \in W$. Moreover, any other direct summand of $\mathrm{BS}(\underline{w})$ is a shift of B_x for some $x < w$. This last statement is equivalent to the fact that B_w has $R_w(-\ell(w))$ exactly once in its Δ-filtration and all the other subquotients are shifts of R_x for $x < w$.

(3) *The character induces an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-modules*

$$\mathrm{ch} : [\mathrm{SBim}]_{\oplus} \xrightarrow{\sim} H(W, S),$$

which is reciprocal to c.

(4) *(Soergel's Hom formula). For $B, B' \in \mathrm{SBim}$, the graded R-bimodule $\mathrm{Hom}^{\bullet}(B, B')$ is free of finite rank as a left R-module and as a right R-module. Moreover, the graded ranks of $\mathrm{Hom}^{\bullet}(B, B')$ as a left and as a right R-module are both equal to*

$$\underline{\mathrm{rk}}(\mathrm{Hom}^{\bullet}(B, B')) = (\mathrm{ch}(B), \mathrm{ch}(B')),$$

where $(-,-) : H(W, S) \times H(W, S) \to \mathbb{Z}[v^{\pm 1}]$ is the usual pairing on the Hecke algebra.

Example 3.13. *By the first and third points in the theorem, if $\underline{w} = (s, t, \ldots, u)$ is an expression for $w \in W$, then we have*

$$\mathrm{ch}(\mathrm{BS}(\underline{w})) = \underline{H}_s \underline{H}_t \cdots \underline{H}_u.$$

Using Soergel's Hom formula, we can compute that

$$\underline{\mathrm{rk}}(\mathrm{Hom}^{\bullet}(B_s, B_t)) = v^2, \quad \underline{\mathrm{rk}}(\mathrm{Hom}^{\bullet}(B_s, B_s)) = v^2 + 1,$$
$$\underline{\mathrm{rk}}(\mathrm{Hom}^{\bullet}(B_s^2, B_s)) = v^3 + 2v + v^{-1}, \quad \underline{\mathrm{rk}}(\mathrm{Hom}^{\bullet}(B_s, B_s B_t B_s)) = v^4 + 2v^2 + 1.$$

Conjecture 3.14 (Soergel's conjecture). *For any $w \in W$, we have $\mathrm{ch}(B_w) = \underline{H}_w$.*

This is now a theorem of Elias and Williamson.

This conjecture implies the Kazhdan-Lusztig conjecture. Recall that the elements $\underline{H}_x$ of the Kazhdan-Lusztig basis are uniquely written as $\underline{H}_x = H_x + \sum_{y<x} h_{y,x} H_y$, where $h_{y,x} \in \mathbb{Z}[v^{\pm 1}]$ are the *Kazhdan-Lusztig polynomials*. The Kazhdan-Lusztig conjecture states that $h_{y,x} \in \mathbb{N}[v^{\pm 1}]$, i.e. that $h_{y,x}$ all have non-negative coefficients. Soergel's conjecture may be rephrased as $h_{y,x} = h_y(B_x)$ and this, together with the fact that $h_x(B) \in \mathbb{N}[v^{\pm 1}]$ for any $B \in \mathrm{SBim}$, indeed implies the Kazhdan-Lusztig conjecture. There is no known proof of the Kazhdan-Lusztig conjecture that does not use categorification. However, for a finite Weyl group, this conjecture was proved to hold using the following result:

Theorem 3.15 (Kazhdan-Lusztig). *For $x, y \in W$ with $y < x$, let $IH^i_{X_y}(X_x) := \mathbb{H}^i(IC_x)_y$, where X_x and X_y are the Schubert varieties associated to x and y, respectively. Then, we have*

$$v^{\ell(x)-\ell(y)} h_{y,x}(v^{-1}) = \sum_i v^i \dim(IH^{2i}_{X_y}(X_x)).$$

Part 4. Relation with Lusztig's multiplicity conjecture

7. Reminders on Lusztig's conjecture

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, with triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and let $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$. For $\lambda \in \mathfrak{h}^*$, we have the Verma module $\Delta(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one-dimensional representation of $\mathfrak{h}$ with weight λ. There exists a unique simple quotient $L(\lambda)$ of $\Delta(\lambda)$, which is finite-dimensional if and only if λ is regular dominant. The category $\mathcal{O}$ (of locally $\mathfrak{n}^+$-finite finitely generated weight modules) has a "block decomposition" $\mathcal{O} = \sum_{\lambda \in \mathfrak{h}^*/(W,\cdot)} \mathcal{O}_\lambda$, where $\cdot$ is the dot action $w \cdot \mu := w(mu + \rho) - \rho$ where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. The *principal block* is $\mathcal{O}_0$ and we have a bijection

$$\begin{array}{ccc} W & \xleftrightarrow{1:1} & \mathrm{Irr}(\mathcal{O}_0) \\ w & \longmapsto & L(w \cdot 0) = L(w\rho - \rho). \end{array}$$

For simplicity we denote $\Delta_w := \Delta(w \cdot 0)$ and $L_w := L(w \cdot 0)$ for any $w \in W$. The Lusztig multiplicity conjecture is stated as follows:

Conjecture 4.1 (Lusztig)**.** *For any $x, y \in W$, we have*

$$[\Delta_y : L_x] = h_{y,x}(1),$$

where $[\Delta_y : L_x]$ is the Jordan-Hölder multiplicity of L_x in Δ_y.

Remark 4.2. *By the Bernstein-Gelfand-Gelfand reciprocity, we have $[\Delta_y : L_x] = (P_x : \Delta_y)$, where $(P_x : \Delta_y)$ is the standard multiplicity of Δ_y in the* projective cover *P_x of L_x.*

This conjecture is now a theorem and the first proof of it (Beilinson-Bernstein and Brylinski-Kashiwara, 1981) uses the localization theorem, the Riemann-Hilbert correspondence and the theorem of Kazhdan and Lusztig stated at the end of the last part. The second proof is by Soergel (in 1990) and uses the functor $\mathbb{V}$. In this part, we explain the strategy of Soergel's proof.

8. Reminders on translation functors and wall-crossing functors

Definition 4.3. *(1) For a positive root $\alpha \in \Phi^+$, let H_α be the affine hyperplane*

$$H_\alpha := \{\lambda \in E := \mathbb{R} \otimes_{\mathbb{Z}} X^*(T) \subset \mathfrak{h}^* \ ; \ (\lambda + \rho, \alpha) = 0\}.$$

(2) The (closed) ρ-shifted Weyl chamber *is defined by*

$$\overline{C}_{\mathbb{Z}} := \{\lambda \in E \ ; \ (\lambda + \rho, \alpha_s) \geq 0, \ \forall s \in S\}.$$

(3) For $\lambda \in \mathfrak{h}^*$, consider the natural maps $i_\lambda : \mathcal{O}_\lambda \hookrightarrow \mathcal{O}$ and $p_\lambda : \mathcal{O} \twoheadrightarrow \mathcal{O}_\lambda$. For $\lambda, \mu \in \mathfrak{h}^*$ such that $\mu - \lambda$ is integral, let ν be the unique dominant integral weight in $W(\mu - \lambda)$. The translation functor *is defined by*

$$T_\lambda^\mu := p_\mu \circ (L(\nu) \otimes -) \circ i_\lambda : \mathcal{O}_\lambda \longrightarrow \mathcal{O}_\mu.$$

Lemma 4.4. *Let $\lambda, \mu \in \mathfrak{h}^*$ such that $\mu - \lambda$ is integral.*

(1) The functor T_λ^μ is exact and sends projective objects to projective objects.
(2) We have a biadjunction between T_λ^μ and T_μ^λ.

Theorem 4.5. *If $\lambda, \mu \in \mathfrak{h}^*$ are dominant integral weights, then the functor $T_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ is an equivalence of categories, with inverse T_μ^λ. Moreover, for $w \in W$, the functor T_λ^μ sends $\Delta(w \cdot \lambda)$ to $\Delta(w \cdot \mu)$.*

Definition 4.6. *Let $s \in S$ and choose an integral μ in $\overline{C}_{\mathbb{Z}}$ such that $\mu \in H_{\alpha_s}$ but $\mu \notin H_{\alpha'}$ for $\alpha' \neq \alpha_s$ (i.e.* Stab.$(\mu) = \{1, s\}$*). We define the* wall-crossing functor

$$\Theta_s := T_\mu^0 \circ T_0^\mu : \mathcal{O}_0 \longrightarrow \mathcal{O}_0.$$

It doesn't depend on μ and is self-biadjoint.

Proposition 4.7. *For any $w \in W$ and $s \in S$ we have a non-split short exact sequence*

$$\begin{cases} 0 \longrightarrow \Delta_w \longrightarrow \Theta_s \Delta_w \longrightarrow \Delta_{ws} \longrightarrow 0 & \text{if} \quad ws > w, \\ 0 \longrightarrow \Delta_{ws} \longrightarrow \Theta_s \Delta_w \longrightarrow \Delta_w \longrightarrow 0 & \text{if} \quad ws < w. \end{cases}$$

In particular, in the (non-split) Grothendieck group $K^0(\mathcal{O}_0)$, we have $[\Theta_s \Delta_w] = [\Delta_w] + [\Delta_{ws}]$, so that the map

$$\begin{array}{ccc} K^0(\mathcal{O}_0) & \longrightarrow & \mathbb{Z}[W] \\ [\Delta_w] & \longmapsto & w \end{array}$$

is an isomorphism of abelian groups intertwining Θ_s and the right multiplication by $1 + s \in \mathbb{Z}[W]$.

Remark 4.8. *This results says that the principal block $\mathcal{O}_0$, together with the family $\{\Theta_s\}_{s \in S}$ categorifies the right regular $\mathbb{Z}[W]$-module.*

The following result is a general fact on finite length abelian categories with enough projectives:

Lemma 4.9. *The family $\{P_w\}_{w\in W}$ is a complete set of non-isomorphic indecomposable projective objects in $\mathcal{O}_0$ and for any projective object $Q \in \mathcal{O}_0$, we have*

$$Q \simeq \bigoplus_{w\in W} P_w^{\oplus \mathrm{mult}(P_w,Q)},$$

where $\mathrm{mult}(P_w, Q) = \dim \mathrm{Hom}(Q, L_w)$.

For any expression $\underline{x} = (s, t, \ldots, u)$, we let

$$P_{\underline{x}} := \Theta_u \circ \cdots \circ \Theta_t \circ \Theta_s(P_1).$$

The following result is a first analogy we can notice between the present setting and Soergel bimodules:

Proposition 4.10. *For $x \in W$, the projective module P_x admits a standard filtration such that Δ_y appears as a subquotient only if $y \leq x$ and Δ_x appears exactly once. Moreover, if $\underline{x}$ is a reduced expression for x, then we have*

$$P_{\underline{x}} \simeq P_x \oplus \bigoplus_{y<x} P_y^{\oplus m_y},$$

for some $m_y \in \mathbb{N}$.

Corollary 4.11. *For $x \in W$ and for a reduced expression $\underline{x}$ of x, the module P_x is the unique indecomposable direct summand of $P_{\underline{x}}$ not appearing as a direct summand of $P_{\underline{w}}$ for any expression $\underline{w}$ with $\ell(\underline{w}) < \ell(\underline{x})$.*

9. Soergel modules

Here, we define the Soergel modules in general. Consider a Coxeter system (W, S), together with an arbitrary W-representation $\mathfrak{h}$ over an arbitrary field $\mathbb{F}$ and let $R := \mathrm{Sym}(\mathfrak{h}^*)$, with $\deg(\mathfrak{h}^*) = 2$. As in the second part, we can define the categories $\mathrm{BSBim}(\mathfrak{h}, W)$ of Bott-Samelson bimodules and the category $\mathrm{SBim}(\mathfrak{h}, W)$ of Soergel bimodules. We recall the dependence in $\mathfrak{h}$ in order to recall that these categories do not necessarily verify the results of the second and third part, which are valid only for certain faithful representations of W (see [EMTW20]).

Definition 4.12. *If $\underline{w} = (s, t, \ldots, u)$ is an expression for $w \in W$, we define the* (right) Bott-Samelson module $\overline{\mathrm{BS}}(\underline{w})$ *is the graded right R-module*

$$\overline{\mathrm{BS}}(\underline{w}) := \mathbb{F} \otimes_R \mathrm{BS}(\underline{w}) \simeq \mathbb{F} \otimes_R R \otimes_{R^s} R \otimes_{R^t} \cdots \otimes_{R^u} R(\ell(\underline{w})).$$

A (right) Soergel module *is a direct summand of a finite direct sum of shifts of Bott-Samelson modules. The category $\overline{\mathrm{SBim}}(\mathfrak{h}, W)$ is the full subcategory of $\mathbf{gmod}{-}R$ consisting of Soergel modules.*

The functor $\mathbb{F} \otimes_R - : R{-}\mathbf{gbim} \longrightarrow \mathbf{gmod}{-}R$ is additive and sends $\mathrm{BS}(\underline{w})$ to $\overline{\mathrm{BS}}(\underline{w})$ and hence restricts to a functor

$$\mathbb{F} \otimes_R - : \mathrm{SBim}(\mathfrak{h}, W) \longrightarrow \overline{\mathrm{SBim}}(\mathfrak{h}, W)$$

and $\overline{\mathrm{SBim}}(\mathfrak{h}, W)$ is the Karoubian completion of the essential image of the functor $\mathbb{F} \otimes_R - : \mathrm{SBim}(\mathfrak{h}, W) \to \mathbf{gmod}{-}R$.

Proposition 4.13. *If W is finite and if $\mathfrak{h}$ is* reflection faithful[1]*, then the map*

$$\mathbb{F} \otimes_R \mathrm{Hom}^\bullet_{R-\mathbf{gbim}}(B, B') \longrightarrow \mathrm{Hom}^\bullet_{\mathbf{gmod}-R}(\mathbb{F} \otimes_R B, \mathbb{F} \otimes_R B')$$

is an isomorphism for all $B, B' \in \mathrm{SBim}(\mathfrak{h}, W)$.

[1] i.e. there is a bijection $\{\text{reflections of } W\} \xleftrightarrow{1:1} \{\text{hyperplanes of } \mathfrak{h} \text{ fixed by some } w \in W\}$.

Corollary 4.14. *If W is finite and $\mathfrak{h}$ is reflection faithful, then the modules*

$$\overline{B}_w := \mathbb{F} \otimes_R B_w$$

are indecomposable and $\{\overline{B}_w,\ w \in W\}$ is a complete set of non-isomorphic indecomposable Soergel modules, up to shift.

Remark 4.15. *(1) The hypotheses are necessary.*

(2) The corollary says that we can identify $[\overline{\mathrm{SBim}}(\mathfrak{h}, W)]_\oplus$ with the right regular representation of $H(W, S)$.

(3) We can replace "right" by "left" everywhere above.

10. The functor $\mathbb{V}$ and Soergel's proof of Lusztig's conjecture

Here, we return to the setting where W is the Weyl group of $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. The functor $\mathbb{V}$ establishes an equivalence

$$\mathrm{Proj}(\mathcal{O}_0) \xleftrightarrow{\mathbb{V}} \overline{\mathrm{SBim}}(\mathfrak{h}^*, W).$$

First, we dualize the datum: Bott-Samelson and Soergel (bi)modules are over $R = \mathrm{Sym}(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$. This comes from the fact that Soergel theoretic objects are *naturally* associated to the *Langlands dual* $\mathfrak{g}^\vee$ of $\mathfrak{g}$.

Remark 4.16. *Since the representations $\mathfrak{h}$ and $\mathfrak{h}^*$ are both isomorphic to the complex geometric representation of W, the above discussions on Bott-Samelson and Soergel (bi)modules apply and the corresponding categories for $\mathfrak{h}$ and $\mathfrak{h}^*$ are (not naturally) equivalent.*

Definition 4.17. *Let $w_0 \in W$ be the longest element in W and consider $P_{w_0} := P(w_0 \cdot 0)$ the projective cover of $L(w_0 \cdot 0)$. We define the functor*

$$\mathbb{V} := \mathrm{Hom}_{\mathcal{O}_0}(P_{w_0}, -) : \mathcal{O}_0 \longrightarrow \mathbf{mod}\mathrm{-End}_{\mathcal{O}_0}(P_{w_0}).$$

Recall that, if we let I_W be the homogeneous ideal generated by W-invariant polynomials with zero constant term, then the *coinvariant algebra* of W is the graded algebra $C := R/I_W$.

Theorem 4.18 (Soergels Endomorphismensatz)**.** *Let $\gamma : Z(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h}) \simeq R$ be the Harish-Chandra homomorphism, post-compose it with the map $R \to R$ sending $P(\lambda)$ to $P(\lambda - \rho)$ and still denote the resulting map by $\gamma : Z(\mathfrak{g}) \to R$. Then, the maps $Z(\mathfrak{g}) \xrightarrow{\gamma} R \xrightarrow{can.} C$ and $Z(\mathfrak{g}) \to \mathrm{End}_{\mathcal{O}_0}(P_{w_0})$ are surjective and have the same kernel. In particular, we have an isomorphism*

$$\mathrm{End}_{\mathcal{O}_0}(P_{w_0}) \simeq C.$$

This result allows one to view the functor $\mathbb{V}$ as a functor

$$\mathbb{V} : \mathcal{O}_0 \longrightarrow \mathbf{mod}\mathrm{-}C,$$

where $\mathbf{mod}\mathrm{-}C$ is the category of ungraded right C-modules.

Theorem 4.19 (Soergel)**.** *(1) (Struktursatz). If $M, Q \in \mathcal{O}_0$ with Q projective, then $\mathbb{V}$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_0}(M, Q) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{mod}-C}(\mathbb{V}(M), \mathbb{V}(Q)).$$

In particular, $\mathbb{V}$ is fully faithful on projective objects.

(2) For $s \in S$, there is a natural isomorphism

$$\mathbb{V} \circ \Theta_s \simeq (- \otimes_{C^s} C) \circ \mathbb{V}.$$

Remark 4.20. *(1) Since P_w is the projective cover of L_w, for all $M \in \mathcal{O}_0$, we have*

$$\dim_{\mathbb{C}} \mathbb{V}(M) = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{O}_0}(P_{w_0}, M)) = [M : L_{w_0}].$$

(2) We have

$$\mathbb{V}(\Delta_w) = \mathbb{C} \quad \textit{and} \quad \mathbb{V}(L_w) = \begin{cases} \mathbb{C} & \textit{if } w = w_0, \\ 0 & \textit{otherwise.} \end{cases}$$

Corollary 4.21. *(1) The functor $\mathbb{V}$ restricts to an equivalence of categories*

$$\mathbb{V} : \mathrm{Proj}(\mathcal{O}_0) \xrightarrow{\sim} \{\textit{ungraded Soergel modules}\}.$$

(2) For $s \in S$, we have $\mathbb{V} \circ \Theta_s \simeq (- \otimes_R B_s) \circ \mathbb{V}$ and moreover, for any expression $\underline{x}$ of $x \in W$, we have

$$\mathbb{V}(P_{\underline{x}}) \simeq \mathbb{C} \otimes_R \mathrm{BS}(\underline{x}) \quad \textit{and} \quad \mathbb{V}(P_x) \simeq \mathbb{C} \otimes_R B_x.$$

We are now ready to deduce Lusztig's conjecture from Soergel's conjecture. Recall that Lusztig's conjecture is equivalent to $(P_x : \Delta_y) = h_{y,x}(1)$. Under the group isomorphism $K^0(\mathcal{O}_0) \simeq \mathbb{Z}[W]$, this in turn is equivalent to the following statement:

$$(*) \qquad \forall x \in W,\ [P_x] = \underline{H}_{x|v=1}.$$

Proposition 4.22. *Soergel's conjecture (for the geometric representation of W) implies the Lusztig multiplicity conjecture.*

Proof. We prove $(*)$ by induction on the Bruhat order. This is trivial if $x = 1$. Otherwise, there is some $s \in S$ such that $w := xs < x$. Then, we have

$$\Theta_s P_w \simeq P_x \oplus \bigoplus_{z<x} P_z^{\oplus m_z},\ m_z \in \mathbb{N}.$$

Applying $\mathbb{V}$ to this isomorphism yields $\overline{B}_w B_s \simeq \overline{B}_x \oplus \bigoplus_{z<x} \overline{B}_z^{\oplus m_z}$ and since Soergel modules decompose exactly as Soergel bimodules do, we get

$$B_w B_s \simeq B_x \oplus \bigoplus_{z<x} B_z^{\oplus m_z}.$$

Applying the character homomorphism to this, we obtain

$$\underline{H}_x = \underline{H}_w \underline{H}_s - \sum_{z<x} m_z \underline{H}_z.$$

On the other hand, using the induction hypothesis and the first equality above, we obtain

$$\begin{aligned} \underline{H}_{x|v=1} = \left(\underline{H}_w \underline{H}_s - \sum_{z<x} m_z \underline{H}_z \right)_{|v=1} &= [P_w](1+s) - \sum_{z<x} m_z [P_z] \\ &= [\Theta_s P_w] - \sum_{z<x} m_z [P_z] = [P_x], \end{aligned}$$

as required. □

Remark 4.23. *(1) For W a Weyl group, Soergel proved his conjecture for the geometric representation using the decomposition theorem. This gives a new almost algebraic proof of Lusztig's conjecture.*

(2) Soergel's program is to give an algebraic proof of Lusztig's conjecture. This was completed in 2013 by Elias and Williamson. This proof has a geometric flavour, as they develop a "Hodge theory of Soergel bimodules"...

Part 5. Abstract for the report

The category of Soergel bimodules is an algebraic generalization of the geometric Hecke category. More precisely, if G is a connected reductuve algebraic groups over $\mathbb{C}$, with a Borel subgroup $B < G$ containing a maximal torus T, with associated Weyl group W and if we let $R := \mathrm{Sym}(X^*(T) \otimes \mathbb{Q})$ (with $\deg(X^*(T)) = 2$), then we have the hypercohomology functor

$$D^b_B(G/B, \mathbb{Q}) \xrightarrow{\mathbb{H}^*_B} H^*_B(G/B, \mathbb{Q})\mathbf{-gmod} = (R \otimes_{R^W} R)\mathbf{-gmod} \xrightarrow{\text{proj.}} R\mathbf{-gbim},$$

where $R-\mathbf{gbim}$ is the category of graded R-bimodules. By a theorem of Soergel, the restriction of this functor to the *geometric Hecke category* (the category of semisimple perverse sheaves on G/B) is fully faithful. Its image in $R-\mathbf{gbim}$ is the category of Soergel bimodules.

This still makes sense for any Coxeter system (W, S), with a sufficiently nice faithful reflection representation $\mathfrak{h}$ of W. In this setting, we introduce the category of Bott-Samelson bimodules and a Soergel bimodule is then defined to be a direct summand of a finite direct sum of shifts of a Bott-Samelson bimodule. Such bimodules form an additive monoidal category denoted by SBim.

We review the basic properties of SBim, in particular the Δ-filtrations and the *character*. By Soergel's categorification theorem, the character is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras

$$\begin{array}{cccc} \mathrm{ch} & : & [\mathrm{SBim}]_{\oplus} & \xrightarrow{\sim} & H(W,S) \\ & & [B] & \longmapsto & \mathrm{ch}(B) \end{array}$$

where $H(W, S)$ is the Hecke algebra of (W, S). Then, we state the classification of indecomposable bimodules, which are parametrized by W and we denote by B_w the indecomposable bimodule associated to $w \in W$.

Soergel's conjecture states that we have $\mathrm{ch}(B_w) \in H(W, S)$ is the element of the Kazhdan-Lusztig basis of $H(W, S)$ corresponding to $w \in W$. In the last part, we discuss the relation between this conjecture (now a theorem of Elias and Williamson) and Lusztig's multiplicity conjecture. First, we notice that Soergel's conjecture also implies the Kazhdan-Lusztig positivity conjecture (stating that the Kazdhan-Lusztig polynomials all have non-negative coefficients). First, we introduce the Soergel modules and Soergel's functor $\mathbb{V}$. We review the main features of this functor and in particular, we state a theorem of Soergel stating that $\mathbb{V}$ establishes an equivalence between the category of projective objects in the principal block $\mathcal{O}_0$ of the category $\mathcal{O}$ and the category of ungraded Soergel modules. We finish by deducing Lusztig's conjecture from Soergel's conjecture for the geometric representation of the Weyl group W.

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