FIRST HOCHSCHILD COHOMOLOGY AND COMPLEMENTS THE WEDDERBURN-MALCEV CASE

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Abstract. This work finds its origin in a question asked by Pr. Ivan Marin in 2016 : In a finite dimensional algebra over an algebraically closed field, what can be said about the unicity of the semisimple subalgebra complementing the radical, given by the Wedderburn-Malcev theorem ? We shall investigate this problem here, by tempting to describe the conjugacy classes of thoses complements. The main idea - which was communicated to me by Pr. Alexander Zimmermann - is to use the first Hochschild cohomology group HH^1, which yields some classification analogous to the case of group cohomology. After some first consequences we shall see, with the (crucial) help of two papers by Rolf Farnsteiner (Bielefeld Universität), that there is only one conjugacy class of such semisimple complements.

1. Scheme of investigations

In order to exhibit the similarities between the case of group cohomology and Hochschild cohomology, we shall first focus on the case of split abelian group extensions. More precisely, we want to describe conjugacy classes of complements in a semidirect product. Indeed, we shall see that these conjugacy classes are parametrized by the first cohomology group H^1. To get the desired bijection, one has to look at the bar resolution of the group algebra, which allows us to describe H^1 as the quotient of derivations by inner derivations. This analysis (which we supposed known here, see [12], [11] or [3] for a proof) yields a correspondence between derivations and complements, which turns out to factorize through inner derviations, to give the main result. Then we give a corollary from [8] which says that - with a little solvable assumption - all the complements are conjugate.

After that, we come to the core if this work, which is the case of algebras. The context here is the one of split Hochschild extensions (see [6]). The main idea is the same as the first case : we describe the first Hochschild cohomology HH^1 as a quotient of derivations, find a correspondence between them and the so-called "complements", that factorizes to give the classification result we were looking for. However, we shall see that conjugacy classes have to be taken in a subgroup of the units of the extension, that is proper in general. We will name these conjugacy classes the π-conjugacy classes (or unipotent conjugacy classes).

Next, we shall investigate the case of the Wedderburn-Malcev theorem ([12], Theorem 3.6.9), by applying the results of the second part. The good case is the one of a radical of square zero. In this case, the main result of the previous part is legit and the vanishing of the first Hochschild cohomology group then gives that all the complements are π-conjugate. In the general case, we will give two results on π-conjugacy classes of complements, that are not very powerful, but still may be useful.

At last, we shall correct the third part with an exposition of a very important result (found in [2], Theorem 1), that describes the conjugacy classes in a slightly more general case. We then apply this result to the Wedderburn-Malcev context to finally prove that all the semisimple complements of the radical are unipotently conjugate.

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2. Prerequisites : the case of abelian extensions of groups

Our presentation here is taken from [7].

Let G be a group and M an abelian group that is furthermore a $\mathbb{Z}G$-module. With the action of G on M, one may define the semi-direct product $E := M \rtimes G$ and we get a short exact sequence

$$0 \longrightarrow M \xrightarrow{\iota} R \xrightarrow{\pi} G \longrightarrow 1$$

and we write $M^{\rtimes E}$ for the set of complements of M in E. Of course, any conjugate subgroup of a complement is a complement itself, so E acts on $M^{\rtimes E}$ by conjugation and we write $M_\gamma^{\rtimes E}$ for the set of orbits under action, that is

$$M_\gamma^{\rtimes E} := M^{\rtimes E} \big/ E.$$

Recall that we have (see [3], Proposition 24)

$$H^1(G,M) \overset{\text{def}}{=} \operatorname{Ext}^1_{\mathbb{Z}G}(\mathbb{Z},M) = Z^1(G,M) \Big/ B^1(G,M),$$

where

$$Z^1(G,M) = \mathcal{D}(G,M) = \{f : G \to M \ ; \ f(gh) = gf(h) + f(g)\}$$

and

$$B^1(G,M) = \mathcal{I}(G,M) = \{f : G \to M \ ; \ \exists m \in M \ ; \ f(g) = gm - m\}.$$

First of all, we have

Proposition 1. *There is a natural (set theoretic) bijection*

$$\mathcal{D}(G,M) \approx M^{\rtimes E}.$$

Proof. First of all, one has to note that the choice of an element of $M^{\rtimes E}$ is equivalent to the choice of a section of π. Let $H \in M^{\rtimes E}$ and let $s : G \hookrightarrow E$ be the corresponding section. Since $E = MH$ and $M \cap H = 1$, if $g \in G \hookrightarrow E$, there is a unique pair $(m^{-1}, h) \in M \times H$ such that $g = m^{-1}h$, that is $mg = h$. In particular, for every $g \in G$, there exists a unique $m_g \in M$ such that $m_g g \in H$. Furthermore, if $g, g' \in G$, with corresponding $m_g, m_{g'} \in M$, then one has

$$((g \cdot m_{g'}) \cdot m_g) \cdot gg' = ((g \cdot m_{g'}) \cdot m_g, g)(1, g') = (m_g, g)(m_{g'}, 1)(1, g') = (m_g, g)(m_{g'}, g') \in H$$

and so $m_{gg'} = (g \cdot m_{g'}) \cdot m_g$. That implies that if $\delta_H : g \mapsto m_g$, then $\delta_H(gg') = g\delta_H(g') + \delta_h(g)$ whence $\delta_H \in \mathcal{D}(G,M)$.

Conversely, if $\delta \in \mathcal{D}(G,M)$, then the map

$$\begin{array}{cccc} s & : & G & \hookrightarrow & E \\ & & g & \mapsto & (\delta(g), g) \end{array}$$

is a section of π and hence $H_\delta := s(G) \in M^{\rtimes E}$. We conclude that the map $H \mapsto \delta_H$ is an inverse of $\delta \mapsto H_\delta$ and this one is therefore bijective, as required. $\square$

Choose $H, K \in M^{\rtimes E}$ and suppose they are conjugate, that is $K = H^e = eHe^{-1}$ for $e = (m, g) \in E$. Then $K = m(gHg^{-1})m^{-1}$ and since $g \in E = MH$, there is a $m_0 \in M$ such that $K = H^{m_0}$ and let $m := m_0^{-1}$. For $g \in G$, one may then compute

$$\delta_H(g)g \in H \ \Rightarrow \ (\delta_H(g)g)^{m_0} \in K \ \Rightarrow m^{-1}\delta_H(g)gm \in K \ \Leftrightarrow \ \delta_H(g)m^{-1} \underbrace{gmg^{-1}}_{\in M} g \in K$$

and since M is abelian, this implies

$$\underbrace{(\delta_H(g)gmg^{-1}m^{-1})}_{\in M} g \in K \ \Rightarrow \ \delta_K(g) = \delta_H(g)[g,m] \ \Rightarrow \ (\delta_K - \delta_H)(g) = [g,m] \overset{\text{def}}{=} gm - m,$$

that is $\delta_K - \delta_H \in \mathcal{I}(G,M)$. By reversing this argument, we get that $\delta_K - \delta_H \in \mathcal{I}(G,M)$ implies that $H = K^e$ for some $e \in E$. Finally, we have proven that $\delta_?$ induces a map $\overline{\delta_?} : M_\gamma^{\rtimes E} \to H^1(G,M)$ that is a bijection. We can summarize these considerations with the following result :

Theorem 1. *Let G a group, M a G-module and $E := M \rtimes G$. Then conjugacy classes in E of complements of M are parametrized by the first cohomology $H^1(G,M)$. In other words, there exists a natural bijection*

$$M_\gamma^{\rtimes E} \xrightarrow{\approx} H^1(G,M)\ .$$

We may investigate the case of Hall subgroups, which always admit complements by the Schur-Zassenhaus theorem (see [3] Théorème 10, or [12], Theorem 1.8.47). We will consider here only the case of finite groups. Let G be a finite group and M a finite G-module. We have :

Proposition 2. *One has*

$$|G| \cdot H^1(G,M) = 0 = |M| \cdot H^1(G,M),$$

that is, the order of every element of $H^1(G,M)$ is divisible by $|G|$ and $|M|$. In particular, if $|M|$ and $|G|$ are coprime, then

$$H^1(G,M) = 0.$$

Proof. Let $\delta \in \mathcal{D}(G,M)$. If $g \in G$, by summing over $h \in G$ the relations

$$\delta(gh) = g\delta(h) + \delta(g)$$

one gets

$$|G|\delta(g) = \sum_{h\in G} \delta(gh) - g\sum_{h\in G}\delta(h),$$

and if $m := -\sum_h \delta(h) \in M$, then the previous equation may be written as $|G|\delta(g) = gm - m$ and then $|G|\delta \in \mathcal{I}(G,M)$ and this proves the first equality. Next, if $\delta \in \mathcal{D}(G,M)$ and if $g \in G$, by Lagrange's theorem we have $|M|\delta(g) = 0$ and so $|M|\delta = 0$, hence the result. $\square$

Corollary 1. *If $N \trianglelefteq G$ is a normal abelian Hall subgroup of G, then N has a complement in G and all of these complements are conjugate in G.*

Proof. The fact that N admits a complement is precisely the statement of the theorem of Schur-Zassenhaus. Furthermore, since $N_\gamma^{\rtimes G} \approx H^1\left({}^G/_N, N\right)$, the result follows directly from the previous proposition. $\square$

We may finally give two results from [8], that can also be found in §8 from a graduate course in Group Theory, given by A. Zimmermann in the Université de Picardie Jules Verne.

Lemma 1. *([8], Theorem 5.24)*
If G is a finite solvable group, then every minimal normal subgroup is elementary abelian.

Proof. Let $V \trianglelefteq G$ minimal. If $H \sqsubseteq V$ (which means that H is a characteristic subgroup of V), then $H \trianglelefteq G$ and since V is minimal, this implies $H = 1$ or $H = V$. In particular, for $H = V'$ (the derived subgroup), then one has $V' = 1$ or $V' = V$ and since G is solvable, V is solvable too and so $V' = 1$, that is V is abelian. Hence every Sylow subgroup of V is characteristic in V, so V is an abelian p-group. Then $\{x \in V\ ;\ x^p = 1\} \sqsubseteq V$, therefore V is elementary abelian. $\square$

Theorem 2. *([8], Theorem 7.42)*
Let N be a normal Hall subgroup of G. If either N or G/N is solvable, then all complements of N in G are conjugate.

Proof. Let's write $m := |N|$ and $n := [G:N]$ and let K_1, K_2 be complements of N.

- Suppose that N is solvable. One has $N' \sqsubseteq N \trianglelefteq G$, whence $N' \trianglelefteq G$ and
$$K_1N'/N' = K_1\Big/(K_1 \cap N') = K_1, \text{ since } K_1 \cap N' \leq K_1 \cap N = 1,$$
whence $[K1N' : N'] = n$. Since N is solvable, one has $N' \lneq N$. If $N' = 1$, then N is abelian and this situation has already been investigated before. Else, one has $[G : N'] < |G|$ and an immediate induction on $|G|$ shows that K_iN'/N' $(i = 1, 2)$ are conjugate in G/N :
$$\exists \overline{g} \in G/N \ ; \ \overline{g}\left(K_1N'/N'\right)\overline{g}^{-1} = K_2N'/N' \ \Rightarrow \ gK_1g^{-1} \leq K_2N'$$
and since $N' \neq N$ we get $|K_1N'| < |G|$ and so gK_1g^{-1} in conjugate to K_2 in K_2N', and in G by induction hypothesis.
- Suppose that G/N is solvable. We shall proceed by induction on $|G|$. Let M/N be a minimal normal subgroup of G/N. Since $n \leq M$, one has
$$M = M \cap G = M \cap K_iN = (M \cap K_i)N, \ i = 1, 2$$
and we have $M \cap K_i \trianglelefteq K_i$. By the previous lemma, the solvability of G/N implies that M/N admits a p-subgroup (elementary abelian) for some prime p. If $M = G$, then G/N is itself a p-group (because of the minimality assumption) and hence ths subgroups K_i are Sylow p-subgroups of G and therefore, are conjugate. One may then suppose that $M \lneq G$. Since $M = (M \cap K_i)N$ and $(M \cap K_i) \cap N < K_i \cap N = 1$, the subgroups $M \cap K_i$ are complements of N in M. By induction assumption, there exists $x \in M \leq G$ such that $M \cap K_1 = x(M \cap K_2)x^{-1} = M \cap xK_2x^{-1}$ and, remplacing K_2 by xK_2x^{-1} if necessary, we may suppose that $M \cap K_1 = M \cap K_2 =: J \trianglelefteq K_i$ and this last equation implies that $K_i \leq N_G(J)$. One also has
$$N_G(J) = N_G(J) \cap NK_i = (N_G(J) \cap N)K_i,$$
and
$$J(N_G(J) \cap N) \cap K_i = J(N_G(J) \cap N \cap K_i) = J.$$
Hence, K_i/J are complements of $J(N_G(J) \cap N)/J$ in $N_G(J)/J$. By induction assumption, there exists $\overline{y} \in N_G(J)/J$ such that $K_1/J = \overline{y}\left(K_2/J\right)\overline{y}^{-1}$ and so $K_1 = yK_2y^{-1}$, as was to be shown.

□

Remark 1. *By the celebrated Feit-Thompson Theorem (which must not be lighlty used, since its proof is 300 pages long, and calls out some very elaborated and complicated concepts) says that every finite group of odd order is solvable (it is equivalent to say that every non-trivial finite simple group is of even order). Since the order and the index of N are coprime, at least one of them is odd and so the solvability hypothesis in the previous theorem is always satisfied. At last, the complements of a normal Hall subgroup are always conjugate.*

3. The case of Algebras : First Hochschild Cohomology

We shall first set up the context. Let K be a field, A a K-algebra and M a A-bimodule (that is, a module over the envelopping algebra $A^e \stackrel{\text{def}}{=} A \otimes_K A^{op}$). Recall that (see [6]) a Hochschild extension of A by M is a short sequence

$$0 \longrightarrow M \xrightarrow{\iota} B \xrightarrow{\pi} A \longrightarrow 0$$

where B is an algebra, π is an algebra epimorphism that is K-splits, ι is a monomorphism of K-vector spaces such that $\operatorname{im}(\iota) = \ker(\pi)$ and M is a square zero two-sided ideal of B. Moreover, the condition $M^2 = 0$ allows to write

$$\begin{cases} \iota(\pi(b)m) = bm, \\ \iota(m\pi(b)) = mb, \ \forall (m, b) \in M \times B. \end{cases}$$

Furthermore, two such extensions $B \stackrel{\pi}{\to} A$ and $B' \stackrel{\pi'}{\to} A$ are said to be equivalent if there is a morphism of algebras $f : B \to B'$ (which is automatically an isomorphism) such that the following diagramm commutes :

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\iota"] \arrow[d, equal] & B \arrow[r, "\pi"] \arrow[d, "f"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, "\iota'"] & B' \arrow[r, "\pi'"] & A \arrow[r] & 0 \end{tikzcd}$$

Consider, for an algebra A and M an A-bimodule, the Hochschild cohomology

$$HH^n(A, M) := \operatorname{Ext}^n_{A^e}(A, M), \ \forall n \geq 0.$$

By looking at the bar resolution of M over A, and writing

$$\begin{cases} Z^1(A, M) := \{g \in \operatorname{Hom}_K(A, M) \ ; \ g(ab) = ag(b) + g(a)b\} \\ B^1(A, M) := \{g \in \operatorname{Hom}_K(A, M) \ ; \ \exists m \in M \ ; \ g(a) = am - ma\} \end{cases}$$

as well as

$$\begin{cases} Z^2(A, M) := \{g \in \operatorname{Hom}_K(A \otimes A, M) \ ; \ ag(b \otimes c) - g(ab \otimes c) + g(a \otimes bc) - g(a \otimes b)c = 0\} \\ B^2(A, M) := \{g \in \operatorname{Hom}_K(A \otimes A, M) \ ; \ \exists h \in \operatorname{Hom}_K(A, M) \ ; \ g(a \otimes b) = ah(b) - h(ab) + h(a)b\} \end{cases}$$

one has the following isomorphisms

$$HH^k(A, M) \simeq Z^k(A, M) \Big/ B^k(A, M), \ k = 1, 2.$$

The fundamental theorem about the relationship between Hochschild cohomology and extensions is the following (see [11], Classification Theorem 9.3.1 and [3], Théorème 11) :

Theorem 3. *The equivalence classes of Hochschild extensions of A by M are in $1-1$ correspondence with the second cohomology group $HH^2(A, M)$.*

Finally, given A, a bimodule M and a cocycle $g \in Z^2(A, M)$, one may define a structure of algebra on $B := M \times A$, written as $B := (M \rtimes_g A, +, \cdot_g)$ as follows

$$(m, a) \cdot_g (n, b) := (mb + an + g(a \otimes b), ab).$$

Let $B := M \rtimes_0 A$ be the trivial Hochschild extension (which is isomorphic to $M \rtimes_h A$ for every $h \in B^2(A, M)$) and note that the choice of a coboundary $g \in B^2(A, M)$ is equivalent to the one of a section (of algebras) σ of π. More precisely, on can embed A in B via

$$\begin{array}{cccc} \sigma & : & A & \hookrightarrow & B \\ & & a & \mapsto & (-h(a), a) \end{array}$$

for $h \in \operatorname{Hom}_K(A, M)$ realizes g as a 2-coboundary. So far, we can already notice some similarities with the case of groups, and this will go intensify.

Definition 1. *A* <u>*complement*</u> *Q of M in $B = M \rtimes_0 A$ is a K-subalgebra of B which is a complement of M in B as vector spaces : $B = M \oplus Q$. This implies that $Q \simeq {}^{B}/_{M}$, as K-algebras. Indeed, the injecton $Q \hookrightarrow B$ as well as the natural projection $B \twoheadrightarrow {}^{B}/_{M}$ are algebra morphisms, and the composed morphism is an isomorphism of vector spaces ; hence is an isomorphism of algebras. Note that the assertion $Q \in M^{\rtimes B}$ is equivalent to ask Q to be $\sigma(A)$ for some section σ. We denote by $M^{\rtimes B}$ for the set of such complements.*

Observe that if $Q \in M^{\rtimes B}$, then M is provided with a structure of Q-bimodule. Indeed, if $m \in M$ and $q \in Q \hookrightarrow B$, written as $q = n + a$ with $a \in A$, since M is 2-nilpotent, one may define $qm := am$ and $mq := ma$.

Let $Q \in M^{\rtimes B}$ and $a \in A \overset{\sigma}{\hookrightarrow} B$, there exists a unique pair $(m_a, q_a) \in B$ such that $a = m_a + q_a$ and let $\delta_Q(a) := m_a$. Next, if $a = m_a + q_a$ and $b = m_b + q_b$, then the condition $M^2 = 0$ implies

$$ab = (m_a + q_a)(m_b + q_b) = m_a m_b + m_a q_b + q_a m_b + q_a q_b = \underbrace{m_a q_b + q_a m_b}_{\in M} + \underbrace{q_a q_b}_{\in Q},$$

hence

$$\delta_Q(ab) = m_a q_b + q_a m_b = \delta_Q(a) q_b + q_a \delta_Q(b) = \delta_Q(a)(m_b + q_b) + (m_a + q_a)\delta_Q(b) = \delta_Q(a) b + a\delta_Q(b).$$

This shows that $\delta_Q : A \to M$ is an element of $Z^1(A, M)$.

Next, if $\delta \in Z^1(A, M)$, let $Q_\delta := \{\delta(a) - a,\ a \in A\}$. Since δ is K-linear, Q_δ is a subspace of B and as $M^2 = 0$, one has

$$(\delta(a) - a)(\delta(b) - b) = \delta(a)\delta(b) - \delta(a)b - a\delta(b) + ab = -(\delta(ab) - ab) \in Q_\delta$$

and so Q_δ is a subalgebra of B. Moreover, if $x = m - a \in B$, then $x = \underbrace{(m - \delta(a))}_{\in M} + \underbrace{(\delta(a) - a)}_{\in Q_\delta}$, whence $B = M + Q_\delta$ and since $A \cap M = 0$, one has $Q_\delta \cap M = 0$, so $B = M \oplus Q_\delta$ and we conclude that $Q_\delta \in M^{\rtimes B}$.

Furthermore, if $Q \in M^{\rtimes B}$ and $a \in A$, then

$$a = \delta_Q(a) + q \ \Rightarrow\ \delta_Q(a) - a = -q \in Q \ \Rightarrow\ Q_{\delta_Q} \subset Q$$

and since Q and Q_δ are (vector) complements of M in B, this implies $Q = Q_{\delta_Q}$. If $\delta \in Z^1(A, M)$, we have

$$B \ni a = \delta_{Q_\delta}(a) + q = \delta_{Q_\delta}(a) - \delta(u) + u,$$

and since $a - u \in M \cap A$ and $\delta_{Q_\delta}(a) - \delta(u) \in M \cap A$, one has $a = u$ and $\delta_{Q_\delta}(a) = \delta(u) = \delta(a)$ so $\delta_{Q_\delta} = \delta$.

Hence, the maps $\delta \mapsto Q_\delta$ and $Q \mapsto \delta_Q$ are mutually inverse, whence the following

Proposition 3. *There is a natural bijection $\delta_?$:*

$$Z^1(A, M) \approx M^{\rtimes B}.$$

We shall now go further on this analysis, following the ideas of the first part.

Suppose that $P, Q \in M^{\rtimes B}$ and that there exists $m \in M$ such that

$$Q = (1 + m)P(1 - m) = (1 + m)P(1 + m)^{-1} =: P^{(1+m)}.$$

Let $a \in A$ written as $a = n + q$ with $n \in M$ and $q \in Q$. One can choose $p \in P$ such that

$$a = n + q = n + (1 + m)p(1 - m) = n + p + mp - pm - mpm.$$

Since M is a two-sided ideal, we have $mp, pm, mpm \in M$ and $mpm = 0$ because $M^2 = 0$. Hence

$$a = \underbrace{n + mp - pm}_{\in M} + p \ \Rightarrow\ \delta_P(a) = \delta_Q(a) + mp - pm,$$

and again, because $M^2 = 0$ this implies

$$ma = m(n + mp - pm) + mp \ \Rightarrow \ ma = mp, \text{ and analogously } am = pm.$$

Whence,

$$\delta_P(a) = \delta_Q(a) + ma - am \ \Rightarrow \ (\delta_Q - \delta_P)(a) = am - ma \ \Rightarrow \ \delta_Q - \delta_P \in B^1(A, M).$$

Conversely, if $\overline{\delta_Q} = \overline{\delta_P}$ in $HH^1(A, M)$, then for $q \in Q$, there exists $a \in A$ with

$$q = \delta_Q(a) - a = \delta_P(a) - a + am - ma = p + am - ma = p - (\delta_Q(a) - a)m + m(\delta_P(a) - a)$$
$$= p - qm + mp = (1+m)p - qm \ \Rightarrow \ q(1+m) = (1+m)p \ \Rightarrow \ Q(1+m) = (1+m)P.$$

But, since $m \in M$ and $M^2 = 0$, one has $(1+m)(1-m) = 1 - m^2 = 1 = (1-m)(1+m)$ and so $1+m \in B^\times$ and $(1+m)^{-1} = 1-m$, so the last equation entails $Q = (1+m)P(1+m)^{-1} = P^{(1+m)}$. We finally get that for $P, Q \in M^{\rtimes}$, one has $\overline{\delta_P} = \overline{\delta_Q}$ in $HH^1(A, M)$ if and only if $Q = P^x$ for some $x \in 1 + M$.

Definition 2.
- *It is clear that $1 + M = \pi^{-1}(1)$. Furthermore, this subgroup of $B^\times$ acts on $M^{\rtimes B}$ by conjugation and we denote by $M^{\rtimes B}_{\gamma,\pi}$ the set of orbits under action :*

$$M^{\rtimes B}_{\gamma,\pi} := M^{\rtimes B} \Big/ \pi^{-1}(1).$$

The orbits are called the π-conjugacy classes of complements.
- *The subgroup $\pi^{-1}(1) = 1 + M$ of $B^\times$ may be called the π-unipotent subgroup of the extension, or also the impious subgroup.*

Remark 2. *The name "unipotent" comes from the following example :*
If $\mathcal{T}_2(K)$ denotes the subalgebra of $\mathcal{M}_2(K)$ consisting of upper-triangular matrices and if we consider the two-sided ideal

$$M := \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \ x \in K \right\}$$

then the considered subgroup is

$$1 + M = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \ x \in K \right\} \leq GL_2(K) = \mathcal{M}_2(K)^\times$$

and is formed by unipotent matrices.
Moreover, the name "impious" finds its origin in the fact that, in general, one has $\pi^{-1}(1) \neq B^\times$; which means that one has to restrict the conjugating elements to a proper subgroup of the units of B, in order to get the classification theorem that follows. An example for which we have that the unipotent subgroup is proper is the following :
Choose a field $K \neq \mathbb{F}_2$ and consider the Hochschild extension (where (P) denotes the ideal $P(X)K[X]$ of $K[X]$ generated by P)

$$\begin{tikzcd} 0 \arrow[r] & (X)/(X^2) \arrow[r, "\iota"] & K[X]/(X^2) \arrow[r, "\pi"] & K[X]/(X) \simeq K \arrow[l, bend left, dashed, "\sigma"] \arrow[r] & 0, \end{tikzcd}$$

provided with the canonical section

$$\begin{tikzcd} K \arrow[r, hook] \arrow[rr, bend right, "\sigma"] & K[X] \arrow[r, two heads] & K[X]/(X^2). \end{tikzcd}$$

Therefore, one has $\pi^{-1}(1) = 1 + (X)/(X^2) = 1 + M$ and if $u \in K^\times \setminus \{1\}$, then $(u + (X^2))(u^{-1} + (X^2)) = 1 + (X^2)$ so $\overline{u}$ is a unit of $K[X]/(X^2)$, which is not an element of $\pi^{-1}(1)$, whence $\pi^{-1}(1) \lneq \left(K[X]/(X^2)\right)^\times$.

The previous considerations and definitions may be summed up with :

Theorem 4. *Let A be a K-algebra, M a A-bimodule and consider the trivial Hochschild extension $B := M \rtimes_0 A$ (M is then a 2-nilpotent two-sided ideal of B). Then the π-conjugacy classes of complements of M in B are in $1-1$ correspondence with the first Hochschild cohomology group $HH^1(A,M)$; and the induced map $\overline{\delta_?}$ realizes the bijection. For short, we have*

$$\overline{\delta_?} : M^{\rtimes B}_{\gamma,\pi} \xrightarrow{\approx} HH^1(A,M) \ .$$

4. The Wedderburn-Malcev case

Let K be an field and A a K-algebra. Recall that we define the <u>Jacobson radical</u> $\mathrm{rad}\,(A)$ of A to be the intersection of all maximal left ideals of A. We shall suppose here that A is finite dimensional and that K is algebraically closed. This last hypothesis is too strong and for the Wedderburn-Malcev theorem, it suffices to suppose that $A\big/\mathrm{rad}\,(A)$ is <u>separable</u>, that is, $L \otimes_K \left(A\big/\mathrm{rad}\,(A)\right)$ is semisimple for every field extension L/K (see [12], Remark 3.6.10); but in [3] (Théorème 12), only the case of an algebraically closed field is considered and we shall work in this context. With all these hypothesis, writing $R := \mathrm{rad}\,(A)$, one has

Theorem 5. *(Wedderburn-Malcev, 1942)(see* [3]*, Théorème 12 or* [12]*, Theorem 3.6.9) There exists a semisimple subalgebra S of A, isomorphic to A/R, such that as vector spaces*

$$A = S \oplus R.$$

We want to describe here the other complements of R in A. For this, we first suppose that

$$R^2 = 0,$$

that is, $R = \mathrm{rad}\,(A)$ is nilpotent of order 2. With this assumption, one gets a canonical split Hochschild extension

$$0 \longrightarrow R \longrightarrow A \underset{\leftarrow - -}{\overset{p}{\longrightarrow}} S \longrightarrow 0$$

We are therefore in the context studied below and so we get a bijection

$$R^{\rtimes A}_{\gamma,p} \approx HH^1(S,R) \simeq HH^1\left(A/R, R\right).$$

By the semisimplicity hypothesis, this last group vanishes :

Lemma 2. *One has*

$$HH^1(S,R) = 0.$$

Proof. From the proof of [3], Théorème 12, the envelopping algebra $S^e = \left(A/R\right)^e$ is semisimple and hence, S^e is a projective S-bimodule, so the representable functor

$$\mathrm{Hom}_{S^e}({}_S S_S, ?) \ : \ S\text{--}\mathfrak{Mod}\text{--}S \longrightarrow \mathfrak{Ab}$$

is exact (see [11], Theorem 2.7.6 or [3], Remarque 34 and Lemme 35), it follows that

$$HH^1(S,R) \stackrel{\text{def}}{=} \mathrm{Ext}^1_{S^e}(S,R) = \mathcal{R}^1(\mathrm{Hom}_{S^e}(S,?))(R) = 0,$$

where $\mathcal{R}^k(F)$ denotes the k^{th} right derived functor of the left exact functor F. □

Hence, one gets $R^{\rtimes A}_{\gamma,p} = \left\{ R^{p^{-1}(1)} \right\}$ and so :

Theorem 6. *If A is a finite dimensional algebra over an algebraically closed field K, such that $\mathrm{rad}\,(A)$ is 2-nilpotent, and if $S \simeq A \big/ \mathrm{rad}\,(A)$ is a semisimple subalgebra of A given by the Wedderburn-Malcev theorem (with $\pi : A \twoheadrightarrow S$ the natural projection), then every complement of $\mathrm{rad}\,(A)$ in A, is π-conjugated to S.*

We shall now investigate the general case, that is R is no longer supposed to be 2-nilpotent. We will see that the situation is quite less friendly.

Remark 3. *Indeed, it turns out that the conclusion of Theorem 6 is still true, as described in the last section.*

Firstly, we will give a general result about the π-conjugates of S, with no use of Hochschild theory. We give next another result which uses cohomology and the previous case.

From [12], Lemma 1.6.6, we know that since A is finite dimensional over K, it is an artinian and noetherian algebra, hence its radical R is nilpotent, of nilpotency class $n \geq 1$, say. We shall prove by induction on n the following fact :

Proposition 4. *If T is a complement of R in A, then there exists some $r \in R$ such that*

$$T \subset S^{(1-r)} \oplus R^{n-1}.$$

Proof. If $n = 2$, there is nothing to be shown and so we may suppose that $n \geq 3$. Let

$$\widehat{A} := A \big/ R^{n-1}$$

and

$$\pi : A \twoheadrightarrow \widehat{A}$$

be the natural projection. From [3], Lemme 7, because the algebra $A \big/ R$ is artinian and noetherian, we have

$$\widehat{R} := \mathrm{rad}\,(\widehat{A}) = \mathrm{rad}\,(A) \cdot \widehat{A} = R \cdot A \big/ R^{n-1} = R \big/ R^{n-1}.$$

Let $\widehat{S} := \pi(S)$. One has

$$\widehat{A} = \pi(A) = \pi(S \oplus R) = \pi(S) \oplus \pi(R) = \widehat{S} \oplus \widehat{R} \;\Rightarrow\; \widehat{S} \in \widehat{R}^{\rtimes \widehat{A}}.$$

If $T \in R^{\rtimes A}$, then $\pi(T) \in \widehat{R}^{\rtimes \widehat{A}}$ and by induction assumption ($\widehat{R}^{n-1} = 0$), one has $\pi(T) \subset \widehat{S}^{(1-\widehat{r})} + \widehat{R}^{n-2}$ with $\widehat{r} \in \widehat{R}$. Hence, if $r \in R$ is such that $\pi(r) = \widehat{r}$, then one gets

$$\pi(T) \subset \pi\left(S^{(1-r)} + R^{n-2}\right) \;\Rightarrow\; T \subset S^{(1-r)} + R^{n-2} + R^{n-1} \subset S^{(1-r)} + R^{n-1}$$

and so $T \subset S^{(1-r)} + R^{n-1}$. Furthermore, $S^{(1-r)} \in R^{\rtimes A}$ and so $S^{(1-r)} \cap R^{n-1} \subset S^{(1-r)} \cap R = 0$, whence the result. □

We finally come to another result, that uses cohomology and the 2-nilpotent case. Recall that $R = \mathrm{rad}\,(A)$. On has

Proposition 5. *If T and S are complements of R in A (not necessary semisimple), then there exists $r \in R$ such that*

$$T \oplus R^2 = S^{(1-r)} \oplus R^2.$$

In other words, two complements of R in A are π-conjugate modulo R^2.

Proof. Let

$$\overline{A} := {}^{A}/_{R^2}$$

together with the natural projection $\pi : A \twoheadrightarrow \overline{A}$. Then $\overline{A}$ is artinian and noetherian and as below,

$$\overline{R} := \mathrm{rad}\,(\overline{A}) = \mathrm{rad}\,(A) \cdot \overline{A} = {}^{R}/_{R^2}\,,$$

so $\overline{R}^2 = 0$ and we are in the previous context. If $S, T \in R^{\rtimes A}$, then $\pi(T), \pi(S) \in \overline{R}^{\rtimes \overline{A}}$. Hence, by the 2-nilpotent case, there exists $\overline{r} \in \overline{R}$ such that $\pi(T) = \pi(S)^{(1-\overline{r})}$ and if $r \in R$ verifies $\pi(r) = \overline{r}$, one then gets $\pi(T) = \pi\left(S^{(1-r)}\right)$ and since $T \cap R^2 \subset T \cap R = 0 = S \cap R \supset S \cap R^2$, on has

$$T \oplus R^2 = T + R^2 = S^{(1-r)} + R^2 = S^{(1-r)} \oplus R^2.$$

Of course, we also have an algebra isomorphism $S \simeq T$. □

5. Correction to the Wedderburn-Malcev case : a positive answer by Rolf Farnsteiner

In [2], one may find an analysis of the situation, similar to the one we just made. It turns out that there is a stronger result about π-conjugates of complements : indeed there are all in the same orbit. More precisely, we shall reproduce the result here. Let A be a finite dimensional algebra over a field K and write $R := \mathrm{rad}\,(A)$.

Theorem 7. *([2], Theorem 1)*
If there is a subalgebra S complementing R in A and if T is a separable subalgebra of A, then one may find $r \in R$ with $T^{(1+r)} \subset S$. In other words, S contains at least one π-conjugate of every separable subalgebra of A.

Proof. First suppose that $R^2 = 0$. We shall give two different arguments to prove the result in this particular case.

∗ The context gives rise to a split Hochschild extension

$$0 \longrightarrow R \longrightarrow A \rightleftarrows S \longrightarrow 0\,.$$

Since T is separable, it is semisimple. Indeed, the map $t \mapsto t \otimes 1$ is an isomorphism of algebras $T \simeq T \otimes_K K$ and this last algebra is semisimple. If $T = A$, then A is semisimple, so $R = 0$, in which case there is noting to show. Else, T is contained in a maximal semisimple subalgebra $\overline{T}$ of A, which is a complement of R. Indeed, S is also contained in a maximal semisimple subalgebra $\overline{S}$ of A, that in turn must be a complement of R (because S is a complement) and so $S = \overline{S}$; so all maximal semisimple subalgebras have the same dimension, hence $\overline{T} \cap R = 0$ and $\dim(A) = \dim(R) + \dim(\overline{T})$, so that $\overline{T}$ is a complement. Therefore, replacing T by $\overline{T}$ if necessary, one may suppose that T is itself a complement of R. But since S is semisimple, by Theorem 6, we get that T and S are π-conjugated, hence the result in this case.

∗ We expose here the more elementary argument from [2]. The decomposition $A = S \oplus R$ gives two natural linear maps $f : T \to S$ and $g : T \to R$ such that

$$\forall t \in T,\ t = f(t) + g(t).$$

Direct computations prove that f is a morphism of K-algebras and

$$\forall s, t \in T,\ g(st) = f(s)g(t) + g(s)f(t).$$

Thus f induces a T-bimodule structure on R and under this structure, we have $g \in Z^1(T, R)$. Since T is separable, the envelopping algebra T^e is semisimple. Indeed, let K^a be an algebraic closure of K. By Wedderburn's theorem, one gets

$$T^{op} \otimes_K K^a \simeq \prod_{k=1}^{m} \mathcal{M}_{n_k}(K^a)$$

whence

$$T^e \otimes_K K^a \simeq \prod_{k=1}^{m} T \otimes_K \mathcal{M}_{n_k}(K^a) \simeq \prod_{k=1}^{m} \mathcal{M}_{n_k}(T \otimes_K K^a).$$

Since $T \otimes_K K^a$ is semisimple, each matric ring $\mathcal{M}_{n_k}(T \otimes K^a)$ is also semisimple. Consequently, $T^e \otimes_K K^a$ is semisimple, implying the same property for T^e (see [1], Lemma 2 for this argument). The semisimplicity of T^e entails that

$$HH^1(T, R) = 0.$$

As a result, one can choose an element $r \in R$ such that

$$g(t) = tr - rt = f(t)r - rf(t), \ \forall t \in T.$$

Consequently, keeping in mind that $R^2 = 0$, on gets

$$t = f(t) + g(t) = f(t)(1 + r) - rf(t)$$

so that

$$(1 + r)t(1 + r)^{-1} = (1 + r)t(1 - r) = (1 + r)f(t) - rf(t)(1 - r) = f(t) \in S,$$

for every $t \in T$, proving that $(1 + r)T(1 + r)^{-1} \subset S$.

We now proceed by induction on the nilpotency class $\ell \geq 2$ of R. Consider

$$\widetilde{A} := {}^{A}/_{R^{\ell-1}},$$

as well as $\pi : A \twoheadrightarrow \widetilde{A}$, the natural epimorphism. As before, $\mathrm{rad}\,(\widetilde{A})^{\ell-1} = 0$ and consider the subalgebras $\widetilde{S} := \pi(S)$ and $\widetilde{T} := \pi(T)$. Since $S \simeq {}^{A}/_{R}$ and T is separable, both S and T are semisimple. Hence, one has $S \cap \ker \pi = 0 = T \cap \ker \pi$. Then we have $\widetilde{A} = \widetilde{S} \oplus \mathrm{rad}\,(\widetilde{A})$ and the induction assumption ensures the existence of $m \in R$ such that

$$T^{(1+m)} \subset S \oplus R^{\ell-1} =: B.$$

Thus, $T^{(1+m)}$ is a separable subalgebra of B and $\mathrm{rad}\,(B)^2 = 0$. The case $\ell = 2$ previously investigated provides $n \in \mathrm{rad}\,(A)^{\ell-1}$ such that

$$(1 + n)(1 + m)T(1 + m)^{-1}(1 + n)^{-1} \subset S,$$

and since $(1 + n)(1 + m) = 1 + n + m$, the element $r := n + m \in R$ is the required one. $\square$

We may finally apply this result to the particular case of Wedderburn-Malcev ; and find at last the positive answer we were seeking for :

Corollary 2. *Let A be a finite dimensional algebra over an algebraically closed field K. If S is a semisimple subalgebra of A such that $A = S \oplus \mathrm{rad}\,(A)$ (which exists by Wedderburn-Malcev) and if T is another semisimple algebra completing* $\mathrm{rad}\,(A)$ *in A, then there exists $r \in \mathrm{rad}\,(A)$ such that $T = S^{(1+r)}$.*
In other words, all the semisimple complements of the radical in A are π-conjugate.

Proof. Since T and S are in particular vector complements, they have the same dimension over K, so it is sufficient to prove that $T \subset S^{(1+r)}$ for some $r \in \mathrm{rad}\,(A)$. But as it can be directly seen in the previous proof, instead of the separable assumption on T, we only use the fact that T and T^e are semisimple algebras to get the conclusion of the Theorem. Here, T is supposed to be semisimple and since K is algebraically closed, the Corollary 1.4.17 to the Artin-Wedderburn theorem from [12] ensures that there are some integers $n_1, \ldots, n_m$ such that, as algebras,

$$T \simeq \prod_{i=1}^{m} \mathcal{M}_{n_i}(K),$$

so

$$T^{op} \simeq \prod_{i=1}^{m} \mathcal{M}_{n_i}(K)^{op} \simeq \prod_{i=1}^{m} \mathcal{M}_{n_i}(K^{op}) \simeq \prod_{i=1}^{m} \mathcal{M}_{n_i}(K).$$

From this, tensoring these two expressions over K yields

$$T^e = T \otimes_K T^{op} \simeq \prod_{1 \le i,j \le m} \mathcal{M}_{n_i}(K) \otimes_K \mathcal{M}_{n_j}(K) \simeq \prod_{1 \le i,j \le m} \mathcal{M}_{n_i n_j}(K).$$

Therefore, T^e is semisimple, as a product of semisimple algebras ; hence the result. $\square$

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