ON THE ENDOMORPHISMS OF THE CELLULAR HOMOLOGY COMPLEX $\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R})$ AND ITS GEOMETRIC DECOMPOSITION

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Abstract. We use computer algebra techniques on the endomorphism algebra of the complex $\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R})$ in order to find a *geometric* decomposition of it.

1. Ring theoretic background

Fix a field k and a k-algebra A and recall the following basic facts about local rings and idempotents :

Lemma 1.1. *[DK80, Lemma 3.2.1] If I is a nil-ideal of A, then every idempotent of A/I can be lifted in a unique way into an idempotent of A.*

Theorem 1.2. *[DK80, Theorem 3.2.2]*
The following are equivalent

i) there is a unique maximal (right or left) ideal in A,
ii) the regular (left or right) A-module is indecomposable,
iii) the non-invertible elements of A form a (right or left) ideal,
iv) the algebra $A/\mathrm{rad}\,(A)$ is a division ring.

Under these assumptions, A is said to be a local *algebra.*

Recall further that, in a ring R, a *decomposition of the identity* is a collection $e_1, \dots, e_n$ of idempotents of R such that $e_ie_j = 0$ if $i \neq j$ and $e_1 + \cdots + e_n = 1$.

Proposition 1.3. *[DK80, Theorem 1.7.2]*
For an A-module M, there is a bijective correspondence between decompositions of M as direct sum of submodules and the decompositions of the identity in the algebra $\mathrm{End}_A(M)$.

Corollary 1.4. *An A-module is indecomposable if and only if its endomorphism algebra does not have non-trivial idempotents.*

Corollary 1.5. *For an A-module M, if $\mathrm{End}_A(M)$ is local, then M is indecomposable.*

Lemma 1.6. *(Fitting, 1935) [Zim14, Lemma 1.4.4] Let A be an algebra over a commutative ring, M a noetherian and artinian A-module and $u \in \mathrm{End}_A(M)$. Then, one has a decomposition*

$$M = N \oplus S,$$

where $u_{|N} \in \mathrm{End}\,(N)$ is nilpotent and $u_{|S} \in \mathrm{End}\,(S)$ is an automorphism. Furthermore, one may take $N = \ker(u^n)$ and $S = \mathrm{im}\,(u^n)$ for $n \gg 0$.

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Corollary 1.7. *If M is an indecomposable noetherian and artinian module over A, then every endomorphism of M is either an automorphism or is nilpotent.*

Corollary 1.8. *If M is an indecomposable noetherian and artinian A-module, then* $\mathrm{End}_A(M)$ *is local.*

Corollary 1.9. *Over a noetherian and artinian k-algebra A, a module is indecomposable if and only if its endomorphism algebra is local.*

Remark 1.10. *One may note that every result above (except the first two) generalize to objects in an arbitrary abelian category. We shal apply them in the category of complexes over the right A-modules :* $\mathrm{Ch}_r(A) := \mathrm{Ch}(\mathfrak{Mod} - A)$.

2. Geometric decomposition of $\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R})$

Recall that the cellular homology complex we have found for $\mathfrak{S}_3$ acting on the real points $X(\mathbb{R})$ reads

$$\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R}) = \left(\mathbb{Z}[\mathfrak{S}_3]^4 \xrightarrow{d_3} \mathbb{Z}[\mathfrak{S}_3]^6 \xrightarrow{d_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{d_1} \mathbb{Z}[\mathfrak{S}_3] \right),$$

where

$$d_1 = \begin{pmatrix} 1 - s_\alpha & 1 - s_\beta & 1 - w_0 \end{pmatrix},$$

$$d_2 = \begin{pmatrix} -1 & 1 & 1 & s_\alpha & w_0 - s_\alpha s_\beta & s_\beta - s_\beta s_\alpha \\ s_\beta s_\alpha - s_\beta & s_\alpha - 1 & -w_0 & w_0 & s_\alpha s_\beta & s_\alpha s_\beta \\ s_\beta & s_\beta s_\alpha & s_\alpha - 1 & s_\alpha s_\beta - w_0 & -s_\beta & s_\beta s_\alpha \end{pmatrix},$$

$$d_3 := \begin{pmatrix} 0 & s_\alpha & 0 & 1 \\ -s_\beta s_\alpha & 0 & -w_0 & 0 \\ 0 & s_\beta s_\alpha & 1 & 0 \\ 1 & 0 & 0 & s_\beta s_\alpha \\ -s_\alpha s_\beta & s_\alpha s_\beta & 0 & 0 \\ 0 & 0 & s_\alpha s_\beta & -s_\alpha s_\beta \end{pmatrix}.$$

First, we simplify this complex a little bit. For $\sigma \in \mathfrak{S}_n$ define the permutation matrix $P_\sigma^n := (\delta_{i,\sigma(j)})_{1 \leq i,j \leq n}$ and let

$P := P^6_{(15)(364)}$, $P' := P^4_{(34)}$, $Q := \mathrm{diag}(-s_\beta s_\alpha, -1, s_\beta s_\alpha, s_\beta, -s_\beta, -1)$, $Q' := \mathrm{diag}(w_0, -s_\alpha s_\beta, -1, 1)$.

Then, set

$$d_1' := d_1 = \begin{pmatrix} 1 - s_\alpha & 1 - s_\beta & 1 - w_0 \end{pmatrix},$$

$$d_2' := d_2 P^{-1} Q = \begin{pmatrix} 1 - s_\beta & -1 & w_0 & 1 - w_0 & s_\beta & -1 \\ -1 & 1 - s_\alpha & s_\beta & s_\alpha & 1 - w_0 & w_0 \\ s_\alpha & -s_\beta s_\alpha & 1 - s_\beta & w_0 & -1 & 1 - s_\alpha \end{pmatrix},$$

$$d_3' := Q^{-1}Pd_3P'Q' = \begin{pmatrix} s_\alpha & 1 & 0 & 0 \\ s_\alpha & 0 & 0 & w_0 \\ s_\beta & 0 & -1 & 0 \\ 0 & 0 & w_0 & w_0 \\ 0 & 1 & s_\beta & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix},$$

and

$$\mathcal{K}'_{\mathfrak{S}_3}(\mathbb{R}) = \left(\mathbb{Z}[\mathfrak{S}_3]^4 \xrightarrow{d_3'} \mathbb{Z}[\mathfrak{S}_3]^6 \xrightarrow{d_2'} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{d_1'} \mathbb{Z}[\mathfrak{S}_3] \right).$$

Then, we have an isomorphism $(P'Q', P^{-1}Q, 1, 1) : \mathcal{K}'_{\mathfrak{S}_3}(\mathbb{R}) \xrightarrow{\sim} \mathcal{K}_{\mathfrak{S}_3}(\mathbb{R})$ in $\mathrm{Ch}_r(\mathbb{Z}[\mathfrak{S}_3])$.

Definition 2.1. *For an finite group G, we say that a complex $\mathcal{C}$ in $\mathrm{Ch}_r(\mathbb{Z}[G])$ is* geometric *if all of its homogeneous components $\mathcal{C}_i$ are free $\mathbb{Z}[G]$-modules of finite type over $\mathbb{Z}$. Furthermore, we say that $\mathcal{C} = \mathcal{C}^1 \oplus \mathcal{C}^2$ is a* geometric decomposition *of $\mathcal{C}$, if $\mathcal{C}^1$ and $\mathcal{C}^2$ are geometric and if one of them is homotopy equivalent to zero in $\mathrm{Ch}_r(\mathbb{Z}[G])$. Finally, $\mathcal{C}$ is said ti be* geometrically indecomposable *if no such decomposition exists.*

Remark 2.2. *The interest of such definitions relies in cellular decompositions. Indeed, we see that a cellular homology complex of a free (right) G-space is a geometric complex. Furthermore, if we can build a cellular "sub-decomposition" of the decomposition we started from, then the corresponding complex admits a geometric decomposition. Hence, a geometric decomposition of a cellular homology complex tells us what are the possible properties of a potential cellular decomposition sith less cells. Moreover, over a field k, it is unreasonnable to hope for a cellular homology complex to be indecomposable in $\mathrm{Ch}_r(k[G])$, since the regular module $k[G]$ (hence every free $k[G]$-module of finite dimension over k) splits into irreducible representations of G over k.*

We shall now geometrically decompose $\mathcal{K} := \mathcal{K}'_{\mathfrak{S}_3}(\mathbb{R})$ into geometrically indecomposable summands. To do this, we have to search for idempotents in $\mathrm{End}_{\,\mathrm{Ch}_r(\mathbb{Z}[\mathfrak{S}_3])}(\mathcal{K})$. Since group algebras over rings are nasty, we tensor it with a field k and look for idempotents in $\mathrm{End}_{\,\mathrm{Ch}_r(k[\mathfrak{S}_3])}(k\mathcal{K})$, where $k\mathcal{K} := \mathcal{K} \otimes_{\mathbb{Z}} k$. Then, we try to use the decomposition over $k[\mathfrak{S}_3]$ to build one over $\mathbb{Z}[\mathfrak{S}_3]$. The natural field to look at in the first place is $k = \mathbb{Q}$. The strategy is the following (the code is written at the end of this paper):

1. Implement the algebra $\mathrm{End}_{\,\mathrm{Ch}_r(\mathbb{Q}[\mathfrak{S}_3])}(\mathbb{Q}\mathcal{K})$ on GAP 4 ([GAP19])
2. Seek for idempotents in it (by investigating a basis, or the central primitive idempotents...)
3. For each idempotent $e \in \mathrm{End}_{\,\mathrm{Ch}_r(\mathbb{Q}[\mathfrak{S}_3])}(\mathbb{Q}\mathcal{K})$, compute $\mathcal{K}^0 := \ker(e)$ and $\mathcal{K}^1 := \mathrm{im}\,(e)$, in such a way that $\mathbb{Q}\mathcal{K} = \mathcal{K}^0 \oplus \mathcal{K}^1$,
4. See if the complexes above are geometric and if one of the two is homotopic to zero.
5. Lift, if possible, this decomposition to $\mathrm{Ch}_r(\mathbb{Z}[\mathfrak{S}_3])$.

In fact, in (2) above, we could take an endomorphism u which is non-invertible, as well as $1-u$. Then, since objects have finite length in $\mathbb{Q}[\mathfrak{S}_3]$ (thats why we have to deal with fields), there is an $n \gg 0$ such that for $m \geq n$, one has $\ker(u^m) = \ker(u^n)$ and $\mathrm{im}\,(u^m) = \mathrm{im}\,(u^n)$. Then, we can set $\mathcal{K}^0 := \ker(u^n)$ and $\mathcal{K}^1 := \mathrm{im}\,(u^n)$ in (3).

Define

$$u_3 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & w_0 & s_\alpha s_\beta \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{End}_{\mathbb{Z}[\mathfrak{S}_3]}(\mathbb{Z}[\mathfrak{S}_3]^4), \quad u_2 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -s_\beta & w_0 & 0 & 0 & 0 \\ 0 & s_\beta s_\alpha & -1 & 0 & 0 & 0 \\ 0 & 1 & -s_\alpha s_\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$u := (u_3, u_2, 0, 0) \in \mathrm{End}_{\mathrm{Ch}_r(\mathbb{Z}[\mathfrak{S}_3])}(\mathcal{K}),$$

(it was found in the basis of the endomorphism algebra over $\mathbb{Q}$). Then u is obviously non-invertible, as well as $1-u$ since ${}^t\begin{pmatrix} 0 & 0 & 1+w_0 & 0 \end{pmatrix} \in \ker(1-u_3)$. Observe that

$$u_3^2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & s_\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad u_2^2 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -s_\beta s_\alpha & 1 & 0 & 0 & 0 \\ 0 & s_\beta & -w_0 & 0 & 0 & 0 \\ 0 & s_\alpha & -s_\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence, $e := u^2$ is an idempotent endomorphism and therefore $\mathcal{K} = \ker(e) \oplus \mathrm{im}\,(e)$. Denote e_j^i the "simplified" cells such that

$$\begin{cases} \mathcal{K}_3 = \mathbb{Z}[\mathfrak{S}_3]\left\langle e_1^3, \ldots, e_4^3 \right\rangle \simeq \mathbb{Z}[\mathfrak{S}_3]^4 \\ \mathcal{K}_2 = \mathbb{Z}[\mathfrak{S}_3]\left\langle e_1^2, \ldots, e_6^2 \right\rangle \simeq \mathbb{Z}[\mathfrak{S}_3]^6 \\ \mathcal{K}_1 = \mathbb{Z}[\mathfrak{S}_3]\left\langle e_1^1, e_2^1, e_3^1 \right\rangle \simeq \mathbb{Z}[\mathfrak{S}_3]^3 \\ \mathcal{K}_0 = \mathbb{Z}[\mathfrak{S}_3]\left\langle e^0 \right\rangle \simeq \mathbb{Z}[\mathfrak{S}_3] \end{cases}$$

with $\mathfrak{S}_3$ acting on the right on the cells e_j^i.

Define $\mathcal{K}^k := \ker(e)$ and $\mathcal{K}^i := \mathrm{im}\,(e)$. We see then $\mathcal{K}_3^k = \mathbb{Z}[\mathfrak{S}_3]\left\langle e_1^3, e_2^3, e_3^3 s_\alpha - e_4^3 \right\rangle$ and $\mathcal{K}_2^k = \mathbb{Z}[\mathfrak{S}_3]\left\langle e_1^2, e_4^2, e_5^2, e_6^2, e_2^2 + e_3^2 s_\beta s_\alpha \right\rangle$. Let $f_0^3 := e_3^3 s_\alpha - e_4^3$ and $f_0^2 := e_2^2 + e_3^2 s_\beta s_\alpha$. We compute

$$d_3'(e_3^3) = -e_3^2 - e_4^2 w_0 + e_5^2 s_\beta =: g_0^2$$

and since $u_4^2(e_3^3) = e_3^3$, we obtain

$$\mathcal{K}^i = \left(\mathbb{Z}[\mathfrak{S}_3]\left\langle e_3^3 \right\rangle \longrightarrow \mathbb{Z}[\mathfrak{S}_3]\left\langle g_0^2 \right\rangle \longrightarrow 0 \longrightarrow 0 \right),$$

where the first map sends e_3^3 to g_0^2. In order to get the induced differentials of $\mathcal{K}^k$, we compute

$$d_2'(f_0^2) = d_2'(e_2^2) + d_2'(e_3^2) s_\beta s_\alpha = e_1^1(s_\beta - 1) + e_2^1 - e_3^1 s_\alpha$$

and

$$d_3'(f_0^3) = d_3'(e_3^3) s_\alpha - d_3'(e_4^3) = -f_0^2 w_0 + e_4^2(s_\alpha s_\beta - w_0) + e_5^2 s_\beta s_\alpha + e_6^2$$

and hence

$$\mathcal{K}^k = \left(\mathcal{K}_3^k \xrightarrow{\delta_3'} \mathcal{K}_2^k \xrightarrow{\delta_2'} \mathcal{K}_1 \xrightarrow{\delta_1' = d_1'} \mathcal{K}_0 \right),$$

where $\mathcal{K}_3^k = \mathbb{Z}[\mathfrak{S}_3]\left\langle e_1^3, e_2^3, f_0^3 \right\rangle$, $\mathcal{K}_2^k = \mathbb{Z}[\mathfrak{S}_3]\left\langle e_1^2, f_0^2, e_4^2, e_5^2, e_6^2 \right\rangle$ and

$$\delta_2' = \begin{pmatrix} 1-s_\beta & s_\beta - 1 & 1-w_0 & s_\beta & -1 \\ -1 & 1 & s_\alpha & 1-w_0 & w_0 \\ s_\alpha & -s_\alpha & w_0 & -1 & 1-s_\alpha \end{pmatrix}, \quad \delta_3' = \begin{pmatrix} s_\alpha & 1 & 0 \\ s_\alpha & 0 & -w_0 \\ 0 & 0 & s_\alpha s_\beta - w_0 \\ 0 & 1 & s_\beta s_\alpha \\ 0 & 1 & 1 \end{pmatrix}.$$

Furthermore, the identity of complex $\mathcal{K}^i$ is easily seen to be homotopic to zero, and hence $\mathcal{K}^i$ is homotopy equivalent to zero. Therefore, the decomposition

$$\mathcal{K} = \mathcal{K}^k \oplus \mathcal{K}^i$$

is geometric.

Now, we continue with $\mathcal{K}' := \mathcal{K}^k$ instead of $\mathcal{K}$. Define

$$u_3' := \begin{pmatrix} s_\alpha s_\beta & w_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad u_2' := \begin{pmatrix} s_\beta s_\alpha & 0 & 0 & 0 & 0 \\ s_\beta s_\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$u' := (u_3', u_2', 0, 0)$ and $e' := (u')^3$. Then $(e')^2 = e'$ and

$$(u')_3^3 = \begin{pmatrix} 1 & s_\alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (u')_2^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\mathcal{K}' = \ker(e') \oplus \operatorname{im}(e')$ and, defining $g_1^2 := (e_1^2 + f_0^2)s_\alpha = \delta_3'(e_1^3)$, we get

$$(\mathcal{K}')^i := \operatorname{im}(e') = \Big(\mathbb{Z}[\mathfrak{S}_3]\left\langle e_1^3 \right\rangle \longrightarrow \mathbb{Z}[\mathfrak{S}_3]\left\langle g_1^2 \right\rangle \longrightarrow 0 \longrightarrow 0 \Big),$$

where the first map sends e_1^3 to g_1^2. If we let $f_1^3 := e_1^3 s_\alpha - e_2^3$ and compute

$$\delta_3'(f_1^3) = \delta_3'(e_1^3)s_\alpha - \delta_3'(e_2^3) = f_0^2 - e_5^2 - e_6^2,$$

then we obtain

$$\mathcal{K}'' := (\mathcal{K}')^k := \ker(e') = \left(\mathcal{K}_3'' \xrightarrow{\delta_3''} \mathcal{K}_2'' \xrightarrow{\delta_2''} \mathcal{K}_1 \xrightarrow{\delta_1''=d_1'} \mathcal{K}_0 \right)$$

where $\mathcal{K}_3'' = \mathbb{Z}[\mathfrak{S}_3]\left\langle f_1^3, f_0^3 \right\rangle$, $\mathcal{K}_2'' = \mathbb{Z}[\mathfrak{S}_3]\left\langle f_0^2, e_4^2, e_5^2, e_6^2 \right\rangle$ and

$$\delta_2'' = \begin{pmatrix} s_\beta - 1 & 1 - w_0 & s_\beta & -1 \\ 1 & s_\alpha & 1 - w_0 & w_0 \\ -s_\alpha & w_0 & -1 & 1 - s_\alpha \end{pmatrix}, \qquad \delta_3'' = \begin{pmatrix} 1 & -w_0 \\ 0 & s_\alpha s_\beta - w_0 \\ -1 & s_\beta s_\alpha \\ -1 & 1 \end{pmatrix}.$$

Finally, define

$$u_3'' := \begin{pmatrix} 1 & -w_0 \\ 0 & 0 \end{pmatrix}, \qquad u_2'' := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$e'' := (u_3'', u_2'', 0, 0)$. Then $(e'')^2 = e''$ and we have $\mathcal{K}'' = \ker(e'') \oplus \operatorname{im}(e'')$. Letting $g_2^2 := -f_0^2 w_0 + e_4^2(s_\alpha s_\beta - w_0) + e_5^2 s_\beta s_\alpha + e_6^2 = \delta_3''(f_0^3)$, we get

$$(\mathcal{K}'')^i := \operatorname{im}(e'') = \Big(\mathbb{Z}[\mathfrak{S}_3]\left\langle f_0^3 \right\rangle \longrightarrow \mathbb{Z}[\mathfrak{S}_3]\left\langle g_2^2 \right\rangle \longrightarrow 0 \longrightarrow 0 \Big),$$

where the first map sends f_0^3 to g_2^2. If we let $f_2^3 := f_1^3 + f_0^3 w_0$ and compute

$$\delta_3''(f_2^3) = \delta_3''(f_1^3) + \delta_3''(f_0^3)w_0 = e_4^2(s_\beta - 1) + e_5^2(s_\alpha - 1) + e_6^2(w_0 - 1),$$

then we obtain

$$\mathcal{K}^0 := (\mathcal{K}'')^k := \ker(e'') = \left(\mathcal{K}_3^0 \xrightarrow{\delta_3} \mathcal{K}_2^0 \xrightarrow{\delta_2} \mathcal{K}_1 \xrightarrow{\delta_1=d_1'} \mathcal{K}_0 \right)$$

where $\mathcal{K}_3^0 = \mathbb{Z}[\mathfrak{S}_3]\left\langle f_2^3\right\rangle$, $\mathcal{K}_2^0 = \mathbb{Z}[\mathfrak{S}_3]\left\langle e_4^2, e_5^2, e_6^2\right\rangle$ and

$$\delta_2 = \begin{pmatrix} 1-w_0 & s_\beta & -1 \\ s_\alpha & 1-w_0 & w_0 \\ w_0 & -1 & 1-s_\alpha \end{pmatrix}, \qquad \delta_3 := \begin{pmatrix} s_\beta - 1 \\ s_\alpha - 1 \\ w_0 - 1 \end{pmatrix}.$$

Moreover, we have isomorphisms of complexes

$$\mathcal{K}^i \simeq (\mathcal{K}')^i \simeq (\mathcal{K}'')^i \simeq \mathcal{Z}(3) := \left(\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{1} \mathbb{Z}[\mathfrak{S}_3] \longrightarrow 0 \longrightarrow 0 \right)$$

and hence these complexes are homotopy equivalent to zero (we denote by $\mathcal{Z}(j)$ the complex with $id_{\mathbb{Z}[\mathfrak{S}_3]}$ between degrees j and $j-1$ and 0 everywhere else). The trick of using GAP

to find idempotents doesn't give any new decomposition over $\mathbb{Z}$. The other idea is to try to express the degree 2 differential as a block matrix, using manipulations on rows and columns, in a 'Smith normal form' fashion. Of course the Smith normal form doesn't make sense in $\mathcal{M}_3(\mathbb{Z}[\mathfrak{S}_3])$, but one could use the same manipulations. For short, let

$$L := \begin{pmatrix} 1 & 0 & 0 \\ -1 & -w_0 & 0 \\ -1 & -s_\beta s_\alpha & s_\alpha \end{pmatrix}, \quad C := \begin{pmatrix} 0 & 0 & w_0 \\ 0 & s_\beta & -s_\alpha s_\beta \\ -1 & 1 & -1 \end{pmatrix}$$

and

$$\widetilde{\delta_1} := \delta_1 L^{-1} = \begin{pmatrix} 0 & s_\alpha s_\beta - 1 & s_\alpha - s_\alpha s_\beta \end{pmatrix}, \quad \widetilde{\delta_3} := C^{-1}\delta_3 = \begin{pmatrix} 0 \\ s_\beta s_\alpha - 1 \\ s_\beta s_\alpha - w_0 \end{pmatrix},$$

$$\widetilde{\delta_2} := L\delta_2 C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 - s_\beta s_\alpha + s_\beta & -s_\beta + s_\alpha - s_\alpha s_\beta + 1 \\ 0 & -1 - w_0 & 2 + s_\alpha - s_\alpha s_\beta \end{pmatrix}.$$

Then, we have an isomorphism

$$\begin{tikzcd} \mathbb{Z}[\mathfrak{S}_3] \arrow[r, "\delta_3"] \arrow[d, equal] & \mathbb{Z}[\mathfrak{S}_3]^3 \arrow[r, "\delta_2"] \arrow[d, "C^{-1}"] & \mathbb{Z}[\mathfrak{S}_3]^3 \arrow[r, "\delta_1"] \arrow[d, "L"] & \mathbb{Z}[\mathfrak{S}_3] \arrow[d, equal] \\ \mathbb{Z}[\mathfrak{S}_3] \arrow[r, "\widetilde{\delta_3}"'] & \mathbb{Z}[\mathfrak{S}_3]^3 \arrow[r, "\widetilde{\delta_2}"'] & \mathbb{Z}[\mathfrak{S}_3]^3 \arrow[r, "\widetilde{\delta_1}"'] & \mathbb{Z}[\mathfrak{S}_3] \end{tikzcd}$$

and this list complex is isomorphic to (we abuse the notations)

$$\widetilde{\mathcal{K}^1} \oplus \mathcal{Z}(2),$$

with $\widetilde{\mathcal{K}^1}$ given by

$$\widetilde{\mathcal{K}^1} := \left(\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{\widetilde{\delta_3}} \mathbb{Z}[\mathfrak{S}_3]^2 \xrightarrow{\widetilde{\delta_2}} \mathbb{Z}[\mathfrak{S}_3]^2 \xrightarrow{\widetilde{\delta_1}} \mathbb{Z}[\mathfrak{S}_3] \right)$$

where

$$\widetilde{\delta_1} = \begin{pmatrix} s_\alpha s_\beta - 1 & s_\alpha - s_\alpha s_\beta \end{pmatrix}, \ \widetilde{\delta_2} = \begin{pmatrix} s_\beta - s_\beta s_\alpha - 1 & s_\alpha - s_\beta - s_\alpha s_\beta + 1 \\ -1 - w_0 & 2 + s_\alpha - s_\alpha s_\beta \end{pmatrix}, \ \widetilde{\delta_3} = \begin{pmatrix} s_\beta s_\alpha - 1 \\ s_\beta s_\alpha - w_0 \end{pmatrix}.$$

We can simplify this a bit further : write $P := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ and $Q := \begin{pmatrix} -s_\alpha s_\beta & 0 \\ 0 & 1 \end{pmatrix}$, then we have

$$\begin{tikzcd} \mathbb{Z}[\mathfrak{S}_3] \arrow[r, "\widetilde{\delta_1}"] \arrow[d, "w_0"] & \mathbb{Z}[\mathfrak{S}_3]^2 \arrow[r, "\widetilde{\delta_2}"] \arrow[d, "P"] & \mathbb{Z}[\mathfrak{S}_3]^2 \arrow[r, "\widetilde{\delta_1}"] \arrow[d, "Q"] & \mathbb{Z}[\mathfrak{S}_3] \arrow[d, "s_\alpha"] \\ \mathbb{Z}[\mathfrak{S}_3] \arrow[r, "\delta_3^1"'] & \mathbb{Z}[\mathfrak{S}_3]^2 \arrow[r, "\delta_2^1"'] & \mathbb{Z}[\mathfrak{S}_3]^2 \arrow[r, "\delta_1^1"'] & \mathbb{Z}[\mathfrak{S}_3] \end{tikzcd}$$

and then we let

$$\delta_1^1 := \begin{pmatrix} w_0 - s_\beta & 1 - s_\beta \end{pmatrix}, \ \delta_2^1 := \begin{pmatrix} s_\alpha s_\beta + s_\beta s_\alpha - w_0 & s_\beta - 1 - s_\alpha s_\beta \\ 1 + w_0 & 1 + s_\alpha - s_\alpha s_\beta - w_0 \end{pmatrix}, \ \delta_3^1 := \begin{pmatrix} w_0 - 1 \\ s_\alpha - 1 \end{pmatrix},$$

and

$$\mathcal{K}^1 := \left(\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{\delta_3^1} \mathbb{Z}[\mathfrak{S}_3]^2 \xrightarrow{\delta_2^1} \mathbb{Z}[\mathfrak{S}_3]^2 \xrightarrow{\delta_1^1} \mathbb{Z}[\mathfrak{S}_3] \right).$$

We now claim that the complex $\mathcal{K}^1$ is geometrically indecomposable. To see this, we can have a look at elementary invariants of matrices. Recall that we have the canonical basis $\mathcal{B} := \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, w_0\}$ of $\mathbb{Z}[\mathfrak{S}_3]$ and we denote by $\mathrm{Mat}^\ell(w)$ the matrix of the left multiplication by w in the basis $\mathcal{B}$. For a given homomorphism of right $\mathbb{Z}[\mathfrak{S}_3]$-modules $\mathbb{Z}[\mathfrak{S}_3]^m \to \mathbb{Z}[\mathfrak{S}_3]^n$ if it is represented in the canonical basis $\mathcal{B}$ of $\mathbb{Z}[\mathfrak{S}_3]$ by the matrix $d = (d_{i,j})_{i,j} \in \mathcal{M}_{n,m}(\mathbb{Z}[\mathfrak{S}_3])$ and if we decompose $d_{i,j} := \sum_{w \in \mathfrak{S}_3} d_{i,j}^w w \in \mathbb{Z}[\mathfrak{S}_3]$ in $\mathcal{B}$, then denoting $d_w := (d_{i,j}^w)_{i,j} \in \mathcal{M}_{n,m}(\mathbb{Z})$, then the corresponding homomorphism of abelian groups $\mathbb{Z}^{6m} \to \mathbb{Z}^{6n}$ is represented (in the canonical basis) by the matrix

$$\mathrm{Kro}(d) := \sum_{w \in \mathfrak{S}_3} d_w \otimes_K \mathrm{Mat}^\ell(w) \in \mathcal{M}_{6n,6m}(\mathbb{Z}),$$

where $\otimes_K$ is the Kronecker tensor product of matrices.

Here, it may be computed that the elementary invariants of the matrix $\mathrm{Kro}(\delta_2^1)$ are 1 and 0, appearing five times each, and 2 appearing twice. If one could remove again a geometric factor, it would only be $\mathcal{Z}(2)$ and this would imply that 1 appears at least six times in the invariants of $\mathrm{Kro}(\delta_2^1)$, which is not the case.

Remark 2.3. *It could even be computed that there is exactly one geometric dcomposition of $\mathcal{K}^1$ over $\mathbb{F}_3$ and that no such decomposition exists over $\mathbb{F}_2$. One can also see that there is no complex of the form $\mathbb{F}_2[\mathfrak{S}_3] \to \mathbb{F}_2[\mathfrak{S}_3] \to \mathbb{F}_2[\mathfrak{S}_3] \to \mathbb{F}_2[\mathfrak{S}_3]$ with the required homology.*

In a cohomological purpose, we will just rewrite slightly the complex $\mathcal{K}^0$ above, by exchanging the first two columns of δ_2 (and hence the first two rows of δ_3) and this gives

$$\delta_1 = \begin{pmatrix} 1 - s_\alpha & 1 - s_\beta & 1 - w_0 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} s_\beta & 1 - w_0 & -1 \\ 1 - w_0 & s_\alpha & w_0 \\ -1 & w_0 & 1 - s_\alpha \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} s_\alpha - 1 \\ s_\beta - 1 \\ w_0 - 1 \end{pmatrix}.$$

Remark 2.4. *Using GAP, we may compute the dimension of the endomorphism algebra of the above complexes :*

$$\begin{cases} \dim_k(\mathrm{End}_{\mathrm{Ch}_r(k[\mathfrak{S}_3])}(k\mathcal{K})) = 187 + \delta_{2,\mathrm{char}(k)} \\ \dim_k(\mathrm{End}_{\mathrm{Ch}_r(k[\mathfrak{S}_3])}(k\mathcal{K}')) = 133 + \delta_{2,\mathrm{char}(k)} \\ \dim_k(\mathrm{End}_{\mathrm{Ch}_r(k[\mathfrak{S}_3])}(k\mathcal{K}'')) = 91 + \delta_{2,\mathrm{char}(k)} \\ \dim_k(\mathrm{End}_{\mathrm{Ch}_r(k[\mathfrak{S}_3])}(k\mathcal{K}^0)) = 61 + \delta_{2,\mathrm{char}(k)} \\ \dim_k(\mathrm{End}_{\mathrm{Ch}_r(k[\mathfrak{S}_3])}(k\mathcal{K}^1)) = 31 + \delta_{2,\mathrm{char}(k)} \end{cases}$$

The discussion above leads then to the

Theorem 2.5. *If we denote*

$$\mathcal{Z}(3) := \left(\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{id} \mathbb{Z}[\mathfrak{S}_3] \longrightarrow 0 \longrightarrow 0 \right), \;\; \mathcal{Z}(2) := \left(0 \longrightarrow \mathbb{Z}[\mathfrak{S}_3] \xrightarrow{id} \mathbb{Z}[\mathfrak{S}_3] \longrightarrow 0 \right)$$

and

$$\mathcal{K}^0 := \left(\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{\delta_3} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{\delta_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{\delta_1} \mathbb{Z}[\mathfrak{S}_3] \right),$$

with

$$\delta_1 = \begin{pmatrix} 1-s_\alpha & 1-s_\beta & 1-w_0 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} s_\beta & 1-w_0 & -1 \\ 1-w_0 & s_\alpha & w_0 \\ -1 & w_0 & 1-s_\alpha \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} 1-s_\alpha \\ 1-s_\beta \\ 1-w_0 \end{pmatrix},$$

then we have a geometric decomposition

$$\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R}) \simeq \mathcal{K}^0 \oplus \mathcal{Z}^{\oplus 3}.$$

Furthermore, denoting

$$\mathcal{K}^1 := \left(\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{\delta_3^1} \mathbb{Z}[\mathfrak{S}_3]^2 \xrightarrow{\delta_2^1} \mathbb{Z}[\mathfrak{S}_3]^2 \xrightarrow{\delta_1^1} \mathbb{Z}[\mathfrak{S}_3] \right)$$

with

$$\delta_1^1 := \begin{pmatrix} w_0 - s_\beta & 1-s_\beta \end{pmatrix}, \;\; \delta_2^1 := \begin{pmatrix} s_\alpha s_\beta + s_\beta s_\alpha - w_0 & s_\beta - 1 - s_\alpha s_\beta \\ 1+w_0 & 1+s_\alpha - s_\alpha s_\beta - w_0 \end{pmatrix}, \;\; \delta_3^1 := \begin{pmatrix} w_0 - 1 \\ s_\alpha - 1 \end{pmatrix},$$

then the complex $\mathcal{K}^1$ is geometrically indecomposable and we have

$$\mathcal{K}^0 \simeq \mathcal{K}^1 \oplus \mathcal{Z}(2).$$

In particular, the induced decomposition

$$\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R}) \simeq \mathcal{K}^1 \oplus \mathcal{Z}(2) \oplus \mathcal{Z}(3)^{\oplus 3}$$

is a complete geometric decomposition of the cellular homology complex of $X(\mathbb{R})$.

Corollary 2.6. *With the above notations, one has*

$$\forall i \geq 0, \;\; H_i(\mathcal{K}^1) = H_i(\mathcal{K}^0) = H_i(X(\mathbb{R}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } \; i = 0 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } \; i = 1 \\ 0 & \text{if } \; i = 2 \\ \mathbb{Z} & \text{if } \; i = 3 \\ 0 & \text{if } \; i \geq 4 \end{cases}$$

Moreover, denoting $\mathrm{Ho}_r(\mathbb{Z}[\mathfrak{S}_3])$ the homotopy category of complexes of right $\mathbb{Z}[\mathfrak{S}_3]$-modules, one has isomorphisms of algebras

$$\mathrm{End}_{\,\mathrm{Ho}_r(\mathbb{Z}[\mathfrak{S}_3])}(\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R})) \simeq \mathrm{End}_{\,\mathrm{Ho}_r(\mathbb{Z}[\mathfrak{S}_3])}(\mathcal{K}^j), \;\; \forall j = 0, 1.$$

Corollary 2.7. *If, for a given chain complex $\mathcal{C}$ we write $\mathcal{C}^{op}$ for the cochain complex with homogeneous components $(\mathcal{C}^{op})^i := \mathcal{C}_{-i}$, then the dual complex $(\mathcal{K}^0)^\vee := \mathrm{Hom}\,(\mathcal{K}^0, \mathbb{Z}[\mathfrak{S}_3])$ with differentials $\delta_i^\vee = \mathrm{Hom}_{\,\mathbb{Z}[\mathfrak{S}_3]}(\delta_i, \mathbb{Z}[\mathfrak{S}_3])$ is a cochain complex isomorphic to $(\mathcal{K}^0)^{op}$ and computes the cohomology of $X(\mathbb{R})$.*

Proof. It is known (see [Hat02, Theorem 3.5]) that the cellular cohomology complex is given by the dual of the cellular homology complex. Hence, we only have to show that

$$^t\mathrm{Kro}(\delta_i) = \mathrm{Kro}\left({}^t\delta_i\right), \ \forall i \in \{1,2,3\}.$$

This is a general fact : the above relation holds for any matrix with coefficients in $\mathbb{Z}[G]$ (G a given finite group) if every entry of the matrix is a linear combination of $g \in G$ such that $g^2 = 1$. Indeed, if $\mathrm{Mat}^\ell(g)$ denotes the matrix of the left multiplication by g in the canonical basis of $\mathbb{Z}[G]$, then ${}^t\mathrm{Mat}^\ell(g) = \mathrm{Mat}^\ell(g^{-1})$. Hence, if $M = (m_{i,j})_{i,j} \in \mathcal{M}_{n,m}(\mathbb{Z}[G])$, with $m_{i,j} = \sum_g m_{i,j}^g g$ and $m_g := (m_{i,j}^g)_{i,j} \in \mathcal{M}_{n,m}(\mathbb{Z})$, then, by assumption, $m_{i,j}^g = 0$ if $g^2 \neq 1$ and hence

$$^t\mathrm{Kro}(M) = \sum_{g\in G} {}^t\left(m_g \otimes_K \mathrm{Mat}^\ell(g)\right) = \sum_{g^2=1} {}^tm_g \otimes_K {}^t\mathrm{Mat}^\ell(g) = \sum_g {}^tm_g \otimes_K \mathrm{Mat}^\ell(g) \overset{\mathrm{def}}{=} \mathrm{Kro}\left({}^tM\right).$$

Now, a glance at how the differentials of $\mathcal{K}^0$ were defined in the Theorem 2.5 finishes the proof. $\square$

3. The homology of $X(\mathbb{R})/\mathfrak{S}_3$

We may use the complex $\mathcal{K}^1$ defined above to compute the homology of the quotient space $X(\mathbb{R})/\mathfrak{S}_3$, which is in fact a smooth compact 3-manifold, since the action of $\mathfrak{S}_3$ on $X(\mathbb{R})$ is free.

First, recall some lemmas

Lemma 3.1. *Let G be a finite group, Z a free G-CW-complex and $\pi : Z \twoheadrightarrow Z/G$ the quotient map. If $e \subset Z$ is a cell, then we have a homeomorphism*

$$\pi : e \xrightarrow{\sim} \pi(e).$$

In particular, $\pi(e)$ is a cell in Z/G.

Proof. To fix ideas, we assume that G acts on the right on Z. Define

$$f := \bigcup_{g\in G} eg$$

and $\iota : e \hookrightarrow f$ the natural inclusion. We claim that the following map

$$\begin{array}{rccc} r & : & f & \to & e \\ & & xg\ (x \in e) & \mapsto & x \end{array}$$

is well-defined, continuous and defines a retract of ι. To see that it is well-defined, assume that $xg = yh$ for $x, y \in e$ and for $g, h \in G$. Then $ehg^{-1} \cap e \neq \emptyset$ and since the decomposition is G-cellular, this implies $ehg^{-1} = e$ and, since the action is free, this gives $g = h$ and hence $x = y$. Next, if $U \subset e$ is open, then $r^{-1}(U) = \bigsqcup_g Ug$ is open in f and it is clear that $r \circ \iota = id_e$, and this is our claim.

Now, f is a free G-space with quotient map $\pi_{|f}$, so by the universal property of the quotient, there exists a unique $\overline{r} : f/G \to e$ such that $\overline{r} \circ \pi_{|f} = r$. We may also take $\overline{\iota} := \pi_{|f} \circ \iota$ and we get that $\overline{\iota}$ and $\overline{r}$ are homemorphisms, inverse to each other. Hence, we get homeomorphisms

$$e \overset{\overline{\iota}}{\simeq} f/G = \pi(f) = \bigcup_{g\in G} \pi(eg) = \bigcup_{g\in G} \pi(e) = \pi(e).$$

$\square$

From this, we deduce the

Corollary 3.2. *If G is a finite group and Z is a free G-CW-complex with cells $\{eg,\ e \in E,\ g \in G\}$ (E is a transveral for cells) and $\pi : Z \twoheadrightarrow Z/G$ is the quotient map, then Z/G is also a CW-complex with cells $\{\pi(e),\ e \in E\}$.*

Proposition 3.3. *If G is a finite group, Z is a free G-CW-complex and $\mathcal{K}_\bullet \in \mathrm{Ch}_r(\mathbb{Z}[G])$ is the cellular homology complex of Z, then the induced cellular homology complex of Z/G is given by $\mathcal{K}_\bullet \otimes_{\mathbb{Z}[G]} \mathbb{Z}$.*

Proof. This is obvious, by definition of the induced cellular structure on Z/G. □

Lemma 3.4. *For a finite group G, denote* $\mathrm{aug} : \mathbb{Z}[G] \to \mathbb{Z}$ *the augmentation homomorphism defined by*

$$\mathrm{aug}\left(\sum_{g\in G} a_g g\right) := \sum_g a_g.$$

If we have a homomorphism of right $\mathbb{Z}[G]$-modules $f : \mathbb{Z}[G]^m \to \mathbb{Z}[G]^n$, identified with its matrix in the canonical basis, then the matrix of the induced homomorphism $f \otimes_{\mathbb{Z}[G]} id_{\mathbb{Z}} : \mathbb{Z}^m \to \mathbb{Z}^n$ is given by the matrix $\mathrm{aug}(f)$*, computed term by term.*

Proof. This is straightforward calculation. □

Proposition 3.5. *The cellular homology complex of the induced cellular structure on $X(\mathbb{R})/\mathfrak{S}_3$ is given by*

$$\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R}) \otimes_{\mathbb{Z}[\mathfrak{S}_3]} \mathbb{Z} = \left(\mathbb{Z}^4 \xrightarrow{d_3 \otimes id} \mathbb{Z}^6 \xrightarrow{d_2 \otimes id} \mathbb{Z}^3 \xrightarrow{d_1 \otimes id} \mathbb{Z} \right)$$

where

$$d_1 \otimes id = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix},\ \ d_2 \otimes id = \begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & 1 \end{pmatrix},\ \ d_3 \otimes id = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

In particular, this complex reduces to the simpler one

$$\mathbb{Z} \xrightarrow{\ 0\ } \mathbb{Z} \xrightarrow{\ 2\ } \mathbb{Z} \xrightarrow{\ 0\ } \mathbb{Z}$$

and the homology of $X(\mathbb{R})/\mathfrak{S}_3$ is the given by

$$\forall i \geq 0,\ H_i(X(\mathbb{R})/\mathfrak{S}_3, \mathbb{Z}) = H_i(\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R}) \otimes_{\mathbb{Z}[\mathfrak{S}_3]} \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ \mathbb{Z} & \text{if } i = 3 \\ 0 & \text{if } i \geq 4 \end{cases}$$

Proof. The computation of the differentials $d_i \otimes id$ is direct. Next, the reduction of the complex $\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R})$ to $\mathcal{K}^1$ gives that $\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R})$ reduces to

$$\mathcal{K}^1 \otimes_{\mathbb{Z}[\mathfrak{S}_3]} \mathbb{Z} = \left(\mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \right)$$

and this complex is easily seen to be isomorphic (in $\mathrm{Ch}(\mathbb{Z})$) to

$$\left(0 \longrightarrow \mathbb{Z} \xrightarrow{1} \mathbb{Z} \longrightarrow \mathbb{Z} \right) \oplus \left(\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \right),$$

hence the result. $\square$

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