FUNDAMENTAL DOMAINS FOR EXTENDED AFFINE WEYL GROUPS

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Contents

1. Prerequisites and notations

1.1. **Root data.**

We start by briefly recalling what a *root datum* is and how one can associate a root datum to any connected reductive complex algebraic group (and more specifically to any semisimple compact Lie group). Standard references for what follows are [MT11] and [KJ05].

Definition 1.1.1. ([MT11, Definition 9.10])
A root datum *is a quadruple* $(X, \Phi, Y, \Phi^\vee)$ *where*

(RD1) the elements X and Y are free abelian groups of finite rank, together with a perfect pairing $\langle \cdot, \cdot \rangle : Y \times X \longrightarrow \mathbb{Z}$,

(RD2) the subsets $\Phi \subset X$ and $\Phi^\vee \subset Y$ are (abstract) reduced root systems in $\mathbb{Z}\Phi \otimes_\mathbb{Z} \mathbb{R}$ and $\mathbb{Z}\Phi^\vee \otimes_\mathbb{Z} \mathbb{R}$, respectively,

(RD3) there is a bijection $\Phi \longrightarrow \Phi^\vee$ (denoted by $\alpha \longmapsto \alpha^\vee$) such that $\langle \alpha^\vee, \alpha \rangle = 2$ for every $\alpha \in \Phi$,

(RD4) the reflections s_α of the root system Φ and $s_{\alpha^\vee}$ of $\Phi^\vee$ are respectively given by

$$\forall x \in X, \ s_\alpha(x) := x - \langle \alpha^\vee, x \rangle \alpha$$

and

$$\forall y \in Y, \ s_{\alpha^\vee}(y) := y - \langle y, \alpha \rangle \alpha^\vee.$$

The Weyl group W of the root system Φ (which is isomorphic to the Weyl group of $\Phi^\vee$ via the map $s_\alpha \longmapsto s_{\alpha^\vee}$) is called the Weyl group of the root datum*. Moreover, we say that the root datum $(X, \Phi, Y, \Phi^\vee)$ is* irreducible *if the root system Φ is.*

From this, one can easily define a morphism of root data and the corresponding category of root data.

For a root datum $(X, \Phi, Y, \Phi^\vee)$, we denote by $V := \mathbb{Z}\Phi \otimes_\mathbb{Z} \mathbb{R}$ the ambient space and $V^* := \mathbb{Z}\Phi^\vee \otimes_\mathbb{Z} \mathbb{R}$ (the notation is consistent since $\mathbb{Z}\Phi^\vee \otimes_\mathbb{Z} \mathbb{R}$ may be identified with the dual of $\mathbb{Z}\Phi \otimes_\mathbb{Z} \mathbb{R}$, via the pairing $\langle \cdot, \cdot \rangle$). As usual, we denote by $\Phi^+ \subset \Phi$ the set of positive roots

(with respect to some linear order on V) and by $\Pi \subset \Phi^+$ the corresponding set of simple roots. Recall that there are elements $\varpi_\alpha \in V$ indexed by $\alpha \in \Pi$ such that

$$\forall \beta \in \Pi, \ (\beta^\vee, \varpi_\alpha) = \delta_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$

These elements form a basis of V and are called the *fundamental weights* of Φ. Dually, we can define the *fundamental coweights* $\varpi_\alpha^\vee \in V^*$ of Φ by the property

$$\forall \beta \in \Pi, \ (\varpi_\alpha^\vee, \beta) = \delta_{\alpha,\beta}.$$

We also consider respectively

$$Q := \mathbb{Z}\Phi = \bigoplus_{\alpha \in \Pi} \mathbb{Z}\alpha \subset V \ \text{ and } \ Q^\vee := \mathbb{Z}\Phi^\vee = \bigoplus_{\alpha \in \Pi} \mathbb{Z}\alpha^\vee \subset V^*$$

the *root lattice* and the *coroot lattice* of Φ. Further, we have the respective *weight lattice* and *coweight lattice*:

$$P := (Q^\vee)^\wedge = \{x \in V \ ; \ \forall \alpha \in \Phi, \ \langle \alpha^\vee, x \rangle \in \mathbb{Z}\} = \bigoplus_{\alpha \in \Pi} \mathbb{Z}\varpi_\alpha \subset V \ \text{ and } \ P^\vee := \bigoplus_{\alpha \in \Pi} \mathbb{Z}\varpi_\alpha^\vee \subset V^*.$$

Thus, the abelian group X is a W-lattice between Q and P:

$$Q \subseteq X \subseteq P.$$

If we enumerate the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ (with $n = \mathrm{rk}\,(X) = \dim(V)$) and if $C := (\langle \alpha_i^\vee, \alpha_j \rangle)_{1 \leq i,j \leq n}$ is the Cartan matrix of Φ, then we have

$$\det(C) = [P : Q] = [P^\vee : Q^\vee].$$

Remark 1.1.2. *Note that to give a root datum $(X, \Phi, Y, \Phi^\vee)$ is the same as to give a Euclidean root system Φ together with a W-lattice X such that $Q \subseteq X \subseteq P$. Indeed, given a root datum $(X, \Phi, Y, \Phi^\vee)$, the following bilinear form*

$$(x, y) := \sum_{\alpha \in \Phi} \langle \alpha^\vee, x \rangle \langle \alpha^\vee, y \rangle$$

certainly defines a W-invariant inner product on the ambient space V, makes Φ into a Euclidean root system and X is clearly a lattice in V. On the other hand, given a Euclidean root system and a W-lattice $Q \subseteq \Lambda \subseteq P$, the inner product yields a perfect pairing $\Lambda^\wedge \times \Lambda \longrightarrow \mathbb{Z}$ and then $(\Lambda, \Phi, \Lambda^\vee, \Phi^\vee)$ is indeed a root datum.

The crucial interest of root data relies in the following theorem of Chevalley:

Theorem 1.1.3. (Chevalley classification theorem, [MT11, §9.2])
Let G be a connected reductive algebraic group over an algebraically closed field k and T be a maximal torus of G. Let Φ be the root system associated to the pair (G, T) and denote by $\Phi^\vee := \{\alpha^\vee, \ \alpha \in \Phi\}$ its dual root system. Let moreover $X(T) := \mathrm{Hom}\,(T, \mathbb{G}_m)$ and $Y(T) := \mathrm{Hom}\,(\mathbb{G}_m, T)$ be the character lattice and the cocharacter lattice of T, respectively. Then, $(X(T), \Phi, Y(T), \Phi^\vee)$ is a root datum.
Two semisimple linear algebraic groups are isomorphic if and only if their root data are isomorphic. For each root datum, there is a semisimple algebraic group which realizes it. Finally, the group is simple if and only if the associated root datum is irreducible.

This can also be adapted to the study of Lie groups. As usual, a small gothic letter denotes the Lie algebra of the algebraic (or Lie) group denoted by the same letter, written in capital standard font.

We let G be a semisimple connected algebraic group over $\mathbb{C}$, B be a Borel subgroup of G, K be a compact real form of G (i.e. K is a semisimple compact Lie group such that $\mathfrak{g} = \mathfrak{k} \otimes_\mathbb{R} \mathbb{C}$). Then, the subgroup $T := K \cap B$ is a maximal torus of K and the complexified Lie algebra $\mathfrak{h} := \mathfrak{t} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$.

If Φ denotes the root system of $(\mathfrak{k}, \mathfrak{t})$ (which is just the real form of the root system of $(\operatorname{Lie} g, \mathfrak{h})$) and $\Phi := \{\alpha^\vee, \ \alpha \in \Phi\}$, then $\Phi \subset i\mathfrak{t}^*$ and we may take

$$X(T) := \{d\lambda : \mathfrak{t} \longrightarrow i\mathbb{R}, \ \lambda \in \operatorname{Hom}(T, \mathbb{S}^1)\} \subset i\mathfrak{t}^*$$

the *character lattice* of T and similarly, $Y(T) = X(T)^\wedge := \{x \in i\mathfrak{t} \ ; \ \forall \lambda \in X(T), \ \lambda(x) \in \mathbb{Z}\} \subset i\mathfrak{t}$ is the *cocharacter lattice* of T and the pairing is of course given by

$$X(T) \times Y(T) \ni (\lambda, x) \longmapsto \lambda(x) \in \mathbb{Z}.$$

Then $(X(T), \Phi, Y(T), \Phi^\vee)$ is a root datum. Note that we have isomorphisms $W \simeq N_K(T)/T \simeq N_G(T^\mathbb{C})/T^\mathbb{C}$ and the Killing form $(\cdot, \cdot)$ on $\mathfrak{g}$ restricts to the Killing form on $\mathfrak{k}$ and gives a W-invariant inner product on $V = i\mathfrak{t}^*$.

Notice finally that W acts naturally on T by conjugation by a representative in the normalizer $N_K(T)$. This is well-defined since T is abelian. On the other hand, W acts on V, V^* and on the lattices $X(T)$ and $Y(T)$.

We have the following important result:

Lemma 1.1.4. ([KJ05, Lemma 1])
If $x \longmapsto e^x$ denotes the usual (Lie theoretic) exponential map $\mathfrak{t} \longrightarrow T$, then the normalized *exponential map*

$$\begin{array}{rcl} \exp \ : \ i\mathfrak{t} & \longrightarrow & T \\ x & \mapsto & e^{2i\pi x} \end{array}$$

is surjective and descends to an isomorphism of Lie groups

$$V^*/Y(T) \xrightarrow{\sim} T.$$

Furthermore, this isomorphism is W-equivariant.

We have the following important isomorphisms

(1) $$P/X(T) \simeq \pi_1(K) \quad \text{and} \quad X(T)/Q \simeq Z(K).$$

In particular, one has $|\pi_1(K)| \times |Z(K)| = [P : Q] = \det(C)$.

Theorem 1.1.5. ([KJ05, Theorem 7])
The group K is determined (up to isomorphism) by its Lie algebra $\mathfrak{k}$, the maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and by the lattice $X(T) \subset i\mathfrak{t}^*$, which can be any lattice L such that $Q \subseteq L \subseteq P$.
The group corresponding to $L = P$ is the simply-connected group *and the one associated to $L = Q$ is the* adjoint group.

The previous two results show that the initial problem of finding a W-equivariant cellular structure on the torus T may be reformulated as follows: given a root datum $(X, \Phi, Y, \Phi^\vee)$, with Weyl group W and ambient space $V := \mathbb{Z}\Phi \otimes_\mathbb{Z} \mathbb{R}$, find a W-equivariant cellular structure on the torus V^*/Y.

In this context and, in view of the isomorphisms (1), the group P/X is called the *fundamental group* of the root datum. Notice that this yields a combinatorial way of defining the fundamental group of any connected reductive algebraic group.

Therefore, we shall fix the following notations:

Notation. *Throughout the paper we fix, once and for all, an irreducible root datum $(X, \Phi, Y, \Phi^\vee)$, with ambient space $V = \mathbb{Z}\Phi \otimes \mathbb{R}$, simple roots $\Pi \subset \Phi^+$, Weyl group $W = \langle s_\alpha, \ \alpha \in \Pi \rangle$, fundamental (co)weights $(\varpi_\alpha)_{\alpha \in \Pi}$ and $(\varpi_\alpha^\vee)_{\alpha \in \Pi}$, (co)root lattices Q and $Q^\vee$ and (co)weight lattices P and $P^\vee$, just as in the beginning of this section. Moreover, we consider the natural projection (which is W-equivariant):*

$$\operatorname{pr} \ : V^* \twoheadrightarrow V^*/Y$$

1.2. **Reduction to fundamental domains of (extended) affine Weyl groups.**

For a vector v is some vector space, we denote by t_v the affine endomorphism of the vector space defined by the translation by the vector v. Since X is a W-lattice that stands between Q and P, we may consider the *intermediate affine Weyl group*

$$W_X := t(X) \rtimes W.$$

It is a subgroup of the group $\mathrm{Aff}(V)$ of affine transformations of the space V. In the same way, we may consider the group $W_Y := t(Y) \rtimes W$ of $\mathrm{Aff}(V^*)$ and we have a canonical isomorphism $W_X \simeq W_Y$, induced by the pairing $\langle \cdot, \cdot \rangle$.

In the sequel, by a *fundamental domain* for the action of a (discrete) group G on a topological space Z, we mean a closed connected subspace $\mathcal{F} \subseteq X$ such that:

  * For $1 \neq g \in G$ the subset $\mathcal{F} \cap g\mathcal{F}$ has empty interior.
  * The translates of $\mathcal{F}$ cover the whole space: $Z = \bigcup_{g \in G} g\mathcal{F}$.

Lemma 1.2.1. *If F is a fundamental domain for the action of W_Y on V^*, then the subset $\mathcal{F} := \mathrm{pr}\,(F)$ of V^*/Y is a fundamental domain for the induced action of the Weyl group W. Moreover, if $F \cap (Y \setminus \{0\}) + F) = \emptyset$, then the restricted map*

$$\mathrm{pr}\, : F \longrightarrow \mathcal{F}$$

is a homeomorphism.

Proof. The fact that the W-translates of $\mathcal{F}$ do cover is obvious. Now let $1 \neq w \in W$ and let us prove that $\mathcal{F} \cap w\mathcal{F}$ has no interior. Since the map pr is onto, it suffices to show that $\mathrm{pr}^{-1}(\mathcal{F} \cap w\mathcal{F})$ has empty interior in V^*. We compute

$$\mathrm{pr}^{-1}(\mathcal{F} \cap w\mathcal{F}) = \mathrm{pr}^{-1}(\mathcal{F}) \cap \mathrm{pr}^{-1}(w\mathcal{F}) = (F + Y) \cap (wF + Y)$$

$$= \left(\bigcup_{x \in Y} t_x(F) \right) \cap \left(\bigcup_{y \in Y} t_y(wF) \right) = \bigcup_{x,y \in Y} t_x(F \cap t_{y-x}(wF)).$$

But since W is finite, we have $W \cap t(Y) = 1$ and because $w \neq 1$, we have $w \notin t(Y)$. As F is a fundamental domain for W_Y, given each $y \in Y$, the closed set $F \cap t_y wF$ has empty interior in V^* and well as $t_y(F \cap t_{y-x}(wF))$. Since Y is countable, the conclusion follows from Baire's theorem. The second statement is a straightforward verification. $\square$

Remark 1.2.2. *Of course, the previous result is* auto-dual, *i.e. it still holds if we replace V by V^* and Y by X.*

1.3. **The affine Weyl group.**

Standard references for what follows are [Bou02], [Hum92].

The affine Weyl group is an infinite extension of the Weyl group W that contains more information about the root system Φ than W but still has some of the good properties of W; namely it is still a Coxeter group. One can use the affine Weyl group to determine the subsystems of Φ and to compute the order of W. What follows is quite classical and may be found for example in [Kan01], [Hum92] and [Bou02].

First, recall that the root system Φ admits a *highest root*. That is, there exists a unique root $\alpha_0 \in \Phi$ such that $\alpha \leq \alpha_0$ for every $\alpha \in \Phi$. For this, see [Hum72, §10.4, Lemma A] or [Kan01, §11.2]. Notice that some authors use the notation $\widetilde{\alpha}$ for the highest root, but the we rather choose α_0 for notational purposes.

Given $\alpha \in \Phi$ and $k \in \mathbb{Z}$, define the following hyperplane of V^*:

$$H_{\alpha,k} := \{\lambda \in V^* \,;\, \langle \lambda, \alpha \rangle = k\}.$$

We also consider the orthogonal reflection $s_{\alpha,k}$ with respect to this hyperplane:

$$s_{\alpha,k} := s_\alpha + k t_{\alpha^\vee} : \lambda \mapsto \lambda - (\langle \lambda, \alpha \rangle - k)\alpha^\vee,$$

with $t_{\alpha^\vee}(\lambda) := \lambda + \alpha^\vee$.

Definition 1.3.1. ([Kan01, §11.1])
Define the affine root system *by* $\Phi_a := \Phi \times \mathbb{Z}$, *the* positive affine roots *by* $\Phi_a^+ := \{(\alpha, n) \in \Phi_a ; \ n > 0 \ or \ n = 0, \ \alpha \in \Phi^+\}$ *and the* negative affine roots *by* $\Phi_a^- := \Phi_a \setminus \Phi_a^+$.
If $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ *and if* $\alpha_0 \in \Phi$ *is the highest root of* Φ, *then we consider the (affine) reflections*

$$\forall 1 \le i \le n, \ s_i := s_{\alpha_i, 0} \quad and \quad s_0 := s_{\alpha_0, 1}.$$

Then, the affine Weyl group *is the subgroup of affine transformations* $\mathrm{Aff}(V^*)$ *generated by these reflections:*

$$W_a := \langle s_0, s_1 \ldots, s_n \rangle.$$

Theorem 1.3.2. ([Kan01, §11.3])
We have the following isomorphism

$$W_a \simeq Q^\vee \rtimes W.$$

Furthermore, if $m_{i,j}$ *denotes the order of* $s_i s_j$ *in* W_a *(for* $0 \le i, j \le n$*), then we have a presentation*

$$W_a = \langle s_0, s_1, \ldots, s_n \mid (s_i s_j)^{m_{i,j}} = 1, \ \forall i, j \rangle.$$

In particular, W_a *is a Coxeter group.*

Similarly to the finite case, one defines the *length* of an element of W_a as follows:

$$\forall w \in W_a, \ \ell(w) := \left| \Phi_a^- \cap w\Phi_a^+ \right|.$$

In this context, we have the important notion of an *alcove*.

Definition 1.3.3. ([Kan01, §11.5])
An alcove *is a connected component of the complementary of the reflecting hyperplanes, i.e. a connected component of the set*

$$V^* \setminus \bigcup_{\substack{\alpha \in \Phi^+ \\ k \in \mathbb{Z}}} H_{\alpha, k}$$

Among these, we consider the fundamental alcove*:*

$$\mathcal{A}_0 := \{\lambda \in V^* ; \ \forall \alpha \in \Phi^+, \ 0 < \langle \lambda, \alpha \rangle < 1\}$$
$$= \{\lambda \in V^* ; \ \forall \alpha \in \Pi, \ \langle \lambda, \alpha \rangle > 0 \ and \ \langle \lambda, \alpha_0 \rangle < 1\}$$

A subset of the boundary of $\mathcal{A}_0$ *of the form* $\partial \mathcal{A}_0 \cap H_{\alpha, 0}$ *or* $\partial \mathcal{A}_0 \cap H_{\alpha, 1}$, *for some* $\alpha \in \Phi^+$, *is called a* wall *of* $\mathcal{A}_0$.

Remark 1.3.4. *If we decompose the highest root* $\alpha_0 \in \Phi^+$ *in the basis* Π *as*

$$\alpha_0 = \sum_{\alpha \in \Pi} n_\alpha \alpha,$$

then, it is shown in [Bou02, V, §2.2, Corollaire] that the closure $\overline{\mathcal{A}_0}$ *is the convex polytope given by*

$$\overline{\mathcal{A}_0} = \mathrm{conv}\left(\{0\} \cup \left\{\frac{\varpi_\alpha^\vee}{n_\alpha}\right\}_{\alpha \in \Pi}\right).$$

The importance of the fundamental alcove relies on the following result:

Theorem 1.3.5. ([Hum92, §4.5 and §4.8])
The closure $\overline{\mathcal{A}_0}$ *of the fundamental alcove is a fundamental domain for* W_a *on* V^*. *Moreover, the group* W_a *acts simply transitively on open alcoves.*

Note the analogy with the notion of a chamber. Recall that a *chamber* is a connected component of $V^* \setminus \bigcup_{\alpha \in \Phi^+} H_{\alpha,0}$. A fundamental domain for the action of W on V^* is given by the closure $\overline{\mathcal{C}_0}$ of the *fundamental Weyl chamber* $\mathcal{C}_0$ defined by

$$\mathcal{C}_0 = \{\lambda \in V^* \; ; \; \forall \alpha \in \Pi, \; \langle \lambda, \alpha \rangle > 0\} = \left\{ \sum_{\alpha \in \Pi} \lambda_\alpha \varpi_\alpha \; ; \; \lambda_\alpha > 0 \right\}.$$

This can be found in [Kan01, §5.2].

2. Fundamental domain for the action of W on V^*/Y

2.1. The simply-connected case.
The main result of this section is the following one:

Proposition 2.1.1. *Assume that the irreducible root datum $(X, \Phi, Y, \Phi^\vee)$ is simply-connected (that is, $X = P$ is the weight lattice). Then the set $\mathrm{pr}(\overline{\mathcal{A}_0})$ is a fundamental domain for the action of W on V^*/Y. Moreover the map restricted projection yields a homeomorphism $\mathrm{pr}(\overline{\mathcal{A}_0}) \simeq \overline{\mathcal{A}_0}$.*
In particular, the decomposition

$$V^*/Y = \coprod_{w \in W} w \cdot \mathrm{pr}(\overline{\mathcal{A}_0})$$

is a W-equivariant triangulation of the torus group V^*/Y.

Only the second part of the last statement deserves a proof. First, we need a small technical lemma:

Lemma 2.1.2. *The following holds*

$$\forall \lambda, \lambda' \in \overline{\mathcal{A}_0}, \; \exists w \in W \; ; \; w(\lambda - \lambda') \in \overline{\mathcal{A}_0}.$$

Proof. Let $\mu := \lambda - \lambda'$. Since, $\lambda, \lambda' \in \overline{\mathcal{A}_0}$, we have $-1 \leq \langle \mu, \alpha \rangle \leq 1$ for $\alpha \in \Phi^+$ and this still holds for any root $\alpha \in \Phi$ because $\Phi^- = -\Phi^+$. Thus, by W-invariance of the pairing $\langle \cdot, \cdot \rangle$, we get

$$\forall \alpha \in \Phi, \; \forall w \in W, \; -1 \leq \langle w(\mu), \alpha \rangle \leq 1$$

and in particular, we have also $\langle w(\mu), \alpha_0 \rangle \in [-1, 1]$.
Now, as the closure $\overline{\mathcal{C}_0}$ of the fundamental Weyl chamber is a fundamental domain for the action of W on V^*, there is some $w \in W$ such that $w(\mu) \in \overline{\mathcal{C}_0}$. But we have seen that $-1 \leq \langle w(\mu), \alpha_0 \rangle \leq 1$ and thus $w(\mu) \in \overline{\mathcal{A}_0}$. $\square$

Remark 2.1.3. *The proof above shows in particular that an element $w \in W$ such that $w(\lambda - \lambda') \in \overline{\mathcal{A}_0}$ has length $\ell(w) = |\{\alpha \in \Phi^+ \; ; \; \langle \lambda, \alpha \rangle < \langle \lambda', \alpha \rangle\}|$. Moreover, since the longest element $w_0 \in W$ is characterized by $w_0(\Phi^+) = \Phi^-$, we have $-\mathcal{A}_0 = w_0 \mathcal{A}_0$.*

Proof of proposition 2.1.1. In view of the second part of the Lemma 1.2.1, we have to prove that for $\lambda, \lambda' \in \overline{\mathcal{A}_0}$, we have $\lambda = \lambda'$ as soon as $\lambda - \lambda' \in Q^\vee$. Using the Lemma 2.1.2, we find $w \in W$ such that $w(\lambda - \lambda') \in \overline{\mathcal{A}_0}$. Since W acts on $Q^\vee$, we are left to prove that $\overline{\mathcal{A}_0} \cap Q^\vee = \{0\}$.
Next, recall that the Weyl chamber $\mathcal{C}_0$ is a union of alcoves (see [Hum92, §4.3]) and since W acts simply transitively on the set of chambers, for every alcove $\mathcal{A}$ there is a unique $w \in W$ such that $w\mathcal{A} \subset \mathcal{C}_0$. Hence, if $0 \in \overline{\mathcal{A}}$, then $0 \in \overline{w\mathcal{A}}$ and therefore $w\mathcal{A} = \mathcal{A}_0$ (see [Bou02, VI, §2.2, Proposition 4]). We have shown that for any alcove $\mathcal{A}$ such that $0 \in \overline{\mathcal{A}}$, there exists $w \in W$ such that $\mathcal{A} = w(\mathcal{A}_0)$.
Finally, let $q \in \overline{\mathcal{A}_0} \cap Q^\vee$. Then $\mathrm{t}_{-q}(\mathcal{A}_0)$ is an alcove containing 0 is its closure. The above discussion ensures the existence of $w \in W$ such that $\mathrm{t}_{-q}\mathcal{A}_0 = w\mathcal{A}_0$ and since W_{a} acts simply transitively on the set of alcoves (cf [Kan01, §11.5, Proposition B]), this implies that $\mathrm{t}_q w = 1$ and since $\mathrm{t}(Q^\vee) \cap W = 1$, we get $\mathrm{t}_q = 1$ so $q = 0$. $\square$
We arrive to the main result of this section, which summarizes the discussion above:

Theorem 2.1.4. *Let K be a simply-connected simple compact Lie group, $T \leq K$ a maximal torus, $W := N_K(T)/T$ its Weyl group, $\mathcal{A}_0 \subset i\mathfrak{t}$ the (dual) fundamental alcove and $\exp : i\mathfrak{t} \to T$, $x \mapsto e^{2i\pi x}$ the normalized exponential map. Then $\overline{\mathcal{A}_0}$ is an r-simplex (with $r = \dim(T)$) and its image $\exp(\overline{\mathcal{A}_0}) \subset T$ is a fundamental domain for W, homeomorphic to $\overline{\mathcal{A}_0}$.*

In particular, the decomposition

$$T = \coprod_{w \in W} w \cdot \exp(\overline{\mathcal{A}_0})$$

is a W-equivariant triangulation of the torus T.

2.2. The adjoint case.

We are looking for a fundamental domain for the action of the *extended affine Weyl group* $\widehat{W}_{\mathrm{a}} := \mathrm{t}(P^\vee) \rtimes W$ on V^* in the case where $Y = P^\vee$ is the coweight lattice. Things get trickier in this case, mostly because $\widehat{W}_{\mathrm{a}}$ is no longer a reflection group (except in type B_n as we shall see).

After the study of this question, the author realized that the main result of this section (Corollary 2.2.9 below) was known since 1984 and is due to Komrakov and Premet [KP84]. However, this paper turns out to be rather hard to find on the Internet and is written in Russian. For the sake of self-containment and as our work is independent of the one of Komrakov-Premet, we give a full proof of this. Our method is quite close to the one used in [KP84]. Moreover, we describe the fundamental domain as a convex hull, which is a new result, as far as the author knows.

The group $\widehat{W}_{\mathrm{a}}$ acts on alcoves (transitively since $W_{\mathrm{a}} \trianglelefteq \widehat{W}_{\mathrm{a}}$ does) but not simply-transitively. We introduce the stabilizer

$$\Omega := \{\phi \in \widehat{W}_{\mathrm{a}} \ ; \ \phi \cdot \mathcal{A}_0 = \mathcal{A}_0\}$$

and we see that we have a decomposition $\widehat{W}_{\mathrm{a}} \simeq W_{\mathrm{a}} \rtimes \Omega$. In particular, one has

$$\Omega \simeq \widehat{W}_{\mathrm{a}}/W_{\mathrm{a}} \simeq P^\vee/Q^\vee \simeq P/Q.$$

Thus, Ω is a finite abelian group. The following table details the fundamental groups of the irreducible root systems:

Type	$\Omega \simeq P/Q$
$A_n \ (n \geq 1)$	$\mathbb{Z}/(n+1)\mathbb{Z}$
$B_n \ (n \geq 2)$	$\mathbb{Z}/2\mathbb{Z}$
$C_n \ (n \geq 3)$	$\mathbb{Z}/2\mathbb{Z}$
$D_{2n} \ (n \geq 2)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
$D_{2n+1} \ (n \geq 2)$	$\mathbb{Z}/4\mathbb{Z}$
E_6	$\mathbb{Z}/3\mathbb{Z}$
E_7	$\mathbb{Z}/2\mathbb{Z}$
E_8	1
F_4	1
G_2	1

TABLE 1. Fundamental groups of irreducible root systems

The description of Ω given in [Bou02, VI, §2.3] will be useful. First, to simplify the notations, we denote by $r := \mathrm{rk}(\Phi) = \dim(V)$ the rank of Φ and we index the simple roots by $I := \{1, \ldots, r\}$, that is

$$\Pi = \{\alpha_1, \ldots, \alpha_r\}.$$

Given the highest root $\alpha_0 = \sum_{i=1}^r n_i \alpha_i$ of Φ, recall that a weight ϖ_i is called *minuscule* if $n_i = 1$ and that minuscule weights form a set of representatives of the classes in P/Q (see [Bou02, Chapter VI, Exercise 24]). Dually, we have the same notion and result for *minuscule coweights*. Let

$$J := \{i \in I \ ; \ n_i = 1\}.$$

Proposition-Definition 2.2.1. ([Bou02, VI, §2.3, Proposition 6])
Let $\alpha_0 = \sum_{i \in I} n_i \alpha_i$ be the highest root of Φ and $w_0 \in W$ be the longest element. For $i \in I$, denote by $W_i \leq W$ the Weyl group of the subsystem of Φ generated by $\{\alpha_j \; ; \; j \neq i\} \subset \Pi$. For $i \in J$, let $w_0^i \in W_i$ be the longest element of W_i and $w_i := w_0^i w_0$. Then the element $\mathrm{t}_{\varpi_i^\vee} w_i \in \widehat{W}_\mathrm{a}$ is in Ω and the map

$$\begin{aligned} J &\to \Omega \setminus \{1\} \\ i &\mapsto \omega_i := \mathrm{t}_{\varpi_i^\vee} w_i \end{aligned}$$

is a bijection.

Remark 2.2.2. Note that, with the preceding notations, w_0^i and w_0 have order 2 and $w_i^{-1} = w_0 w_0^i$ sends $\varpi_i^\vee$ to $-\overline{\mathcal{A}_0}$. Indeed, denoting by $\Phi_i := \Phi \cap \mathbb{Z} \langle \Pi \setminus \{\alpha_i\} \rangle$ the subsystem of Φ generated by the simple roots other than α_i, for $j \neq i$ we have $w_0^i \in W(\Phi_i)$ and

$$\langle w_0^i \varpi_i^\vee, \alpha_j \rangle = \langle \varpi_i^\vee, w_0^i \alpha_j \rangle = 0,$$

because $w_0^i \alpha_j \in \Phi_i$. Since $w_0^i \varpi_i^\vee \in P^\vee$ and W stabilizes the set

$$\{\lambda \in V^* \; ; \; \forall \alpha \in \Phi, \; -1 \leq \langle \lambda, \alpha \rangle \leq 1\},$$

we have that $\langle w_0^i \varpi_i^\vee, \alpha_i \rangle$ is an integer between -1 and 1. Of course, it is not 0 and if it is -1, then we have $w_0^i \alpha_i = -\alpha_i$, so $w_0^i(\Pi) \subset \Phi^-$ and thus $w_0^i = w_0$, a contradiction. Hence, we have $w_0^i \varpi_i^\vee = \varpi_i^\vee$ and thus $w_i^{-1} \varpi_i^\vee = w_0 w_0^i \varpi_i^\vee = w_0 \varpi_i^\vee \in -\overline{\mathcal{A}_0}$ since $w_0 \alpha_i \in \Phi^-$.

We shall prove the following result:

Proposition 2.2.3. The subset of $\mathcal{A}_0$ defined by

$$F_{P^\vee} := \{\lambda \in \mathcal{A}_0 \; ; \; \forall i \in J, \; \langle \lambda, \alpha_i + \alpha_0 \rangle \leq 1\}$$

is a fundamental domain for the action of Ω on $\mathcal{A}_0$.

This will be done step-by-step. Once and for all, we fix the notation

$$\alpha_0 = \sum_{i=1}^r n_i \alpha_i$$

for the highest root of Φ. First, we have to obtain more information about how Ω acts on $\mathcal{A}_0$ and how its Weyl part acts on roots. The fact that Ω is formed of affine isomorphisms will considerably constrain these actions.

Lemma 2.2.4. The subgroup $\Omega \leq \widehat{W}_\mathrm{a}$ acts on the set $\{0\} \cup \left\{\frac{\varpi_i^\vee}{n_i}\right\}_{i \in I}$ and on the subset $\{0\} \cup \{\varpi_j^\vee\}_{j \in J}$.

Proof. Let $\omega \in \Omega$. Since $\omega \mathcal{A}_0 = \mathcal{A}_0$ and $\widehat{W}_\mathrm{a} \leq \mathrm{Aff}(V^*)$, we also have $\omega \overline{\mathcal{A}_0} = \overline{\mathcal{A}_0}$. But $\overline{\mathcal{A}_0}$ is a convex polytope with vertices $\{0\} \cup \{\varpi_i^\vee/n_i\}_{i \in I}$ (see [Bou02, VI, §2.2]) and since ω is affine, it sends any vertex of $\overline{\mathcal{A}_0}$ to another one. Hence, Ω indeed acts on the first set.

Now, we have to see why a minuscule coweight $\varpi_j^\vee$ is sent by $\omega \in \Omega$ to 0 or a minuscule coweight. We just have seen that, if it is non-zero, then there is some i such that $\omega(\varpi_j^\vee) = \varpi_i^\vee/n_i$. Using the description given in the Proposition 2.2.1 for Ω, we can write $\omega = \omega_k = \mathrm{t}_{\varpi_k^\vee} w_k$ for some $k \in J$. Note that if $i \in \{j, k\}$, then $n_i = 1$ and thus $\omega(\varpi_j^\vee)$ is indeed a minuscule coweight, so we may assume that $i \neq j, k$. Let $i' \in I$ and consider the simple reflection $s_{i'} : \lambda \mapsto \lambda - \langle \lambda, \alpha_{i'} \rangle \alpha_{i'}^\vee$. We immediately see that $s_{i'}(\varpi_j^\vee) = \varpi_j^\vee - \delta_{i',j} \alpha_j^\vee$ and by induction, we conclude that there are integers $\lambda_{i'}^k \in \mathbb{Z}$ such that

$$w_k(\varpi_j^\vee) = \varpi_j^\vee - \sum_{i'=1}^r \lambda_{i'}^k \alpha_{i'}^\vee.$$

Hence, we compute

$$\frac{\varpi_i^\vee}{n_i} = \omega_k(\varpi_j^\vee) \stackrel{\text{def}}{=} w_k(\varpi_j^\vee) + \varpi_k^\vee = \varpi_j^\vee + \varpi_k^\vee - \sum_{i'=1}^r \lambda_{i'}^k \alpha_{i'}^\vee.$$

Taking the pairing of this equation against α_i yields

$$\frac{1}{n_i} = -\sum_{i'} \lambda_{i'}^k \langle \alpha_{i'}^\vee, \alpha_i \rangle = -\sum_{i'} \lambda_{i'}^k C_{i',i},$$

where $C = (C_{i',i}) \in \mathcal{M}_r(\mathbb{Z})$ is the Cartan matrix. Thus, the sum above is an integer, which implies that $n_i = 1$. $\qquad \square$

Now, we outline the way an element $w_i \in W$ acts on roots.

Lemma 2.2.5. *Let $i \in J$ and consider the corresponding element $\omega_i = t_{\varpi_i^\vee} w_i \in \Omega$. Then the element $w_i^{-1} \in W$ takes $\Pi \setminus \{\alpha_i\}$ into Π and α_i to the lowest root $-\alpha_0$, that is*

$$\begin{cases} w_i^{-1}(\Pi \setminus \{\alpha_i\}) \subset \Pi \\ \quad w_i^{-1}(\alpha_i) = -\alpha_0 \end{cases}$$

Moreover, if $j \neq i$, then $n_{w_i^{-1}\alpha_j} = n_j$; in other words, the numbers of occurrences of α_j and of $w_i^{-1}(\alpha_j)$ in α_0 are the same. In fact, we have

$$\alpha_j = w_i(\alpha_k) \iff \frac{\varpi_j^\vee}{n_j} = \omega_i\left(\frac{\varpi_k^\vee}{n_k}\right).$$

Finally, if $j \in I$ is such that $\omega_i(\varpi_j^\vee) = 0$, then $w_i^{-1}(\alpha_0) = -\alpha_j$.

Proof. First, we prove that $w_i^{-1}(\Pi \setminus \{\alpha_i\}) \subset \Pi$. Consider $\alpha_j \in \Pi$ with $j \neq i$. If we have $\varpi_i^\vee = \omega_i(0) = \frac{\varpi_j^\vee}{n_j}$, then $n_j = 1$ and $i = j$, which is excluded. Thus, by the Lemma 2.2.4, there is some $k \in I$ such that $\omega_i\left(\frac{\varpi_k^\vee}{n_k}\right) = \frac{\varpi_j^\vee}{n_j}$. We compute

$$(*) \qquad \langle \varpi_k^\vee, w_i^{-1}\alpha_j \rangle = n_k \left\langle w_i\left(\frac{\varpi_k^\vee}{n_k}\right), \alpha_j \right\rangle = n_k \left\langle \frac{\varpi_j^\vee}{n_j} - \varpi_i^\vee, \alpha_j \right\rangle = \frac{n_k}{n_j}.$$

Now, let $\ell \neq k$. If $\omega_i\left(\frac{\varpi_\ell^\vee}{n_\ell}\right) = 0$, then we have

$$\langle \varpi_\ell^\vee, w_i^{-1}\alpha_j \rangle = -n_\ell \langle \varpi_i^\vee, \alpha_j \rangle = 0,$$

otherwise, there is some $\ell' \neq j$ such that $\omega_i\left(\frac{\varpi_\ell^\vee}{n_\ell}\right) = \frac{\varpi_{\ell'}^\vee}{n_{\ell'}}$ and we compute similarly

$$\langle \varpi_\ell^\vee, w_i^{-1}\alpha_j \rangle = n_\ell \left\langle w_i\left(\frac{\varpi_\ell^\vee}{n_\ell}\right), \alpha_j \right\rangle = 0.$$

In any case, we have

$$(**) \qquad \qquad \forall \ell \neq k, \ \langle \varpi_\ell^\vee, w_i^{-1}\alpha_j \rangle = 0$$

Hence, equations $(*)$ ans $(**)$ imply that $\Phi \ni w_i^{-1}\alpha_j = \frac{n_k}{n_j}\alpha_k$, thus $n_j = n_k$ and $w_i^{-1}\alpha_j = \alpha_k \in \Pi$. This proves the first and third statements at once.

We have to see why $w_i\alpha_0 = -\alpha_i$. First, since $n_i = 1$, the vector $\varpi_i^\vee$ is a vertex of $\overline{\mathcal{A}_0}$, hence the vector $w_i^{-1}\varpi_i^\vee$ is a vertex of $-\overline{\mathcal{A}_0}$, that is, $w_i^{-1}\varpi_i^\vee$ belongs to the set $\{0\} \cup \{\varpi_j^\vee/n_j\}$. Since $w_i^{-1}\varpi_i^\vee \neq 0$, there is some j such that $w_i^{-1}\varpi_i^\vee = \varpi_j^\vee/n_j$, so we have

$$(\dagger) \qquad \qquad \langle \varpi_i^\vee, w_i\alpha_0 \rangle = \langle w_i^{-1}\varpi_i^\vee, \alpha_0 \rangle = -1.$$

Now, let $j \neq i$. Here again, we have $\omega_i(0) = \varpi_i^\vee \neq \frac{\varpi_j^\vee}{n_j}$, so there exists k such that $\omega_i\left(\frac{\varpi_k^\vee}{n_k}\right) = \frac{\varpi_j^\vee}{n_j}$. We compute, using the equation (†),

(‡)
$$\left\langle \frac{\varpi_j^\vee}{n_j}, w_i\alpha_0 \right\rangle = \left\langle w_i\left(\frac{\varpi_k^\vee}{n_k}\right) + \varpi_i^\vee, w_i\alpha_0 \right\rangle = \left\langle \frac{\varpi_k^\vee}{n_k}, \alpha_0 \right\rangle - 1 = 0.$$

Now, we have $w_i\alpha_0 \in \Phi$ and the equation (‡) ensures that for $j \neq i$, we have $\left\langle \varpi_j^\vee, w_i\alpha_0 \right\rangle = 0$, so there is $\mu \in \{\pm 1\}$ such that $w_i\alpha_0 = \mu\alpha_i$. But if $\mu = 1$, then $w_i^{-1}\alpha_i = \alpha_0 \in \Phi^+$ and using the first statement, we obtain $w_i^{-1}(\Pi) \subset \Phi^+$, thus $w_i = 1$, a contradiction.

To prove the last statement, we compute
$$\left\langle \varpi_j^\vee, w_i^{-1}\alpha_0 \right\rangle = \left\langle w_i\varpi_j^\vee, \alpha_0 \right\rangle = -\left\langle \varpi_i^\vee, \alpha_0 \right\rangle = -1.$$
If $k \neq j$, then there exists ℓ such that $\omega_i\left(\frac{\varpi_k^\vee}{n_k}\right) = \frac{\varpi_\ell^\vee}{n_\ell}$ and we have
$$\left\langle \frac{\varpi_k^\vee}{n_k}, w_i^{-1}\alpha_0 \right\rangle = \left\langle \frac{\varpi_\ell^\vee}{n_\ell} - \varpi_i^\vee, \alpha_0 \right\rangle = 0.$$
These two equations together show that $w_i^{-1}\alpha_0 = -\alpha_j$, as required. $\qquad\square$

Lemma 2.2.6. *For $\omega_i = \mathrm{t}_{\varpi_i^\vee}w_i \in \Omega$ and a simple root $\alpha_k \in \Pi$, the root $w_i\alpha_k$ is simple except in the case where $\omega_i(\varpi_k^\vee) = 0$, in which case it is the lowest root.*

Proof. We compute
$$\omega_i^{-1} = w_i^{-1}\mathrm{t}_{-\varpi_i^\vee} = w_i^{-1}\mathrm{t}_{-\varpi_i^\vee}w_iw_i^{-1} = \mathrm{t}_{-w_i^{-1}(\varpi_i^\vee)}w_i^{-1} = \mathrm{t}_{\omega_i^{-1}(0)}w_i^{-1}.$$

Since Ω acts on $\{0\} \cup \{\varpi_\ell^\vee/n_\ell\}$, there exists $k_0 \in J$ such that $\varpi_{k_0}^\vee = \omega_i^{-1}(0)$ and the conclusion follows from the first two statements of the Lemma 2.2.5. $\qquad\square$

Recall that the *extended Dynkin diagram* of the root system Φ is its Dynkin diagram to which we add a new vertex corresponding to $-\alpha_0$, which is linked to the other vertices in the same way as for the classical Dynkin diagram.

The Lemmae 2.2.5 and 2.2.6 show that the Weyl part of any element of Ω permutes the vertices of the extended diagram and since the pairing is W-invariant, it also preserves the edges of the diagram. Hence we obtain the following result:

Corollary 2.2.7. *The group Ω acts on the extended Dynkin diagram $\widehat{\mathcal{D}}$, giving rise to a normal inclusion $\Omega \trianglelefteq \mathrm{Aut}(\widehat{\mathcal{D}})$. Moreover, the factor group $\mathrm{Aut}(\widehat{\mathcal{D}})/\Omega$ is isomorphic to the automorphism group $\mathrm{Aut}(\mathcal{D}) \leq \mathrm{Aut}(\widehat{\mathcal{D}})$ of the finite Dynkin diagram $\mathcal{D}$ of Φ (see Table 2.2).*
In other words, there is a semi-direct product decomposition
$$\mathrm{Aut}(\widehat{\mathcal{D}}) \simeq \Omega \rtimes \mathrm{Aut}(\mathcal{D}).$$

Proof of Proposition 2.2.3. By definition of a fundamental domain, we have to prove that the set
$$F := \{\lambda \in \mathcal{A}_0 \; ; \; \forall i \in J, \; \langle \lambda, \alpha_0 + \alpha_i \rangle \leq 1\}$$
satisfies the following conditions:
  a) the set F is closed in $\mathcal{A}_0$ and connected,
  b) for $1 \neq \omega \in \Omega$, the set $F \cap \omega F$ has empty interior in $\mathcal{A}_0$,
  c) the union of the Ω-translates of F cover $\mathcal{A}_0$, that is $\mathcal{A}_0 = \bigcup_{\omega \in \Omega} \omega F$.
To prove a), we just have to notice that the closure of F in V^* is an intersection of closed half-spaces, so it is a convex polytope. Thus, the set F, being the intersection of this closure and $\mathcal{A}_0$, is closed and convex.

Let $\omega \in \Omega \setminus \{1\}$. For $\lambda \in F$, writing $\omega = \omega_i = \mathrm{t}_{\varpi_i^\vee}w_i$, we calculate
$$\langle \omega_i(\lambda), \alpha_0 + \alpha_i \rangle = \langle w_i\lambda + \varpi_i^\vee, \alpha_0 + \alpha_i \rangle = 2 + \langle \lambda, w_i^{-1}\alpha_0 + w_i^{-1}\alpha_i \rangle,$$

but, by the Lemma 2.2.5, we have $w_i^{-1}\alpha_i = -\alpha_0$ and there is some $k \in J$ such that $w_i^{-1}\alpha_0 = -\alpha_k \in \Pi$. Since $\lambda \in F$, we have

$$\langle \omega_i(\lambda), \alpha_0 + \alpha_i \rangle = 2 + \langle \lambda, w_i^{-1}\alpha_0 + w_i^{-1}\alpha_i \rangle = 2 - \underbrace{\langle \lambda, \alpha_0 + \alpha_k \rangle}_{\leq 1} \geq 1.$$

This proves that, if $\lambda \in F$, then $\langle \omega_i(\lambda), \alpha_0 + \alpha_i \rangle \geq 1$. Thus, if $\lambda \in F$ and $\omega_i(\lambda) \in F$, then $\langle \omega_i(\lambda), \alpha_0 + \alpha_i \rangle = 1$. Therefore, we have

$$F \cap \omega_i^{-1}F \subseteq \mathcal{A}_0 \cap \{\lambda \in V^* \; ; \; \langle \omega_i(\lambda), \alpha_0 + \alpha_i \rangle = 1\},$$

and the later subset has empty interior in $\mathcal{A}_0$, concluding the proof of b).

Finally, to prove c), let $\lambda \in \mathcal{A}_0$. We have to find some $\omega \in \Omega$ such that $\lambda \in \omega F$. Choose $i \in J$ such that

$$\langle \lambda, \alpha_i \rangle = \max_{j \in J} \langle \lambda, \alpha_j \rangle.$$

If $\langle \lambda, \alpha_0 + \alpha_i \rangle \leq 1$, then $\lambda \in F$ and $\omega = 1$ is the desired element. Otherwise, pick j such that $\omega_i(\varpi_j^\vee) \neq 0$. We have

$$\langle \omega_i^{-1}(\lambda), \alpha_0 + \alpha_j \rangle = \langle w_i^{-1}(\lambda - \varpi_i^\vee), \alpha_0 + \alpha_j \rangle = \langle \lambda - \varpi_i^\vee, w_i\alpha_0 + w_i\alpha_j \rangle = \langle \lambda - \varpi_i^\vee, \alpha_k - \alpha_i \rangle,$$

where $w_i\alpha_j = \alpha_k \in \Pi$ for some k; which is ensured to exist by the Lemma 2.2.6. If $i = k$, then this is zero. Otherwise, we get

$$\langle \omega_i^{-1}(\lambda), \alpha_0 + \alpha_j \rangle = \langle \lambda - \varpi_i^\vee, \alpha_k - \alpha_i \rangle = 1 + \underbrace{\langle \lambda, \alpha_k - \alpha_i \rangle}_{\leq 0} \leq 1.$$

Hence, for every j such that $\omega_i(\varpi_j^\vee) \neq 0$, we have $\langle \omega_i^{-1}(\lambda), \alpha_0 + \alpha_j \rangle \leq 1$. Now, let j such that $\omega_i(\varpi_j^\vee) = 0$. Then, by the Lemma 2.2.6, we have $w_i\alpha_j = -\alpha_0$ and thus

$$\langle \omega_i^{-1}(\lambda), \alpha_0 + \alpha_j \rangle = \langle \lambda - \varpi_i^\vee, w_i\alpha_0 + w_i\alpha_j \rangle = \langle \varpi_i^\vee - \lambda, \alpha_0 + \alpha_i \rangle = 2 - \underbrace{\langle \lambda, \alpha_0 + \alpha_i \rangle}_{>1} < 1.$$

Therefore, for every $j \in J$ we have $\langle \omega_i^{-1}(\lambda), \alpha_0 + \alpha_j \rangle \leq 1$ and thus $\omega_i^{-1}(\lambda) \in F$. $\square$

Remark 2.2.8. *The proof of* c) *above gives a concrete way to find, given $\lambda \in \mathcal{A}_0 \setminus F_{P^\vee}$, an element $\omega_i \in \Omega$ such that $\lambda \in \omega_i F_{P^\vee}$. Namely, it is the element corresponding to the minuscule coweight $\varpi_i^\vee$ with index i such that $\langle \lambda, \alpha_i \rangle = \max_{j \in J} \langle \lambda, \alpha_j \rangle$.*

Corollary 2.2.9. *The convex polytope of V^* defined by*

$$F_{P^\vee} := \{\lambda \in \overline{\mathcal{A}_0} \; ; \; \langle \lambda, \alpha_0 + \alpha \rangle \leq 1, \; \forall \alpha \in \Pi \; ; \; n_\alpha = 1\}$$
$$= \{\lambda \in V^* \; ; \; \langle \lambda, \alpha_0 \rangle \leq 1, \; \forall \alpha \in \Pi, \; \langle \lambda, \alpha \rangle \geq 0 \;\; and \;\; n_\alpha = 1 \;\Rightarrow\; \langle \lambda, \alpha_0 + \alpha \rangle \leq 1\}$$

is a fundamental domain for the extended affine Weyl group $\widehat{W}_\mathrm{a}$.

Proof. This is obvious using the Proposition 2.2.3, the fact that $\overline{\mathcal{A}_0}$ is a fundamental domain for W_a and since $\widehat{W}_\mathrm{a} = W_\mathrm{a} \rtimes \Omega$. $\square$

We can now investigate the injectivity condition. We have the following result:

Lemma 2.2.10. *The hypothesis of the Lemma 1.2.1 is fulfilled in the adjoint case. In other words, one has*

$$F_{P^\vee} \cap (P^\vee \setminus \{0\} + F_{P^\vee}) = \emptyset.$$

Proof. Let $\lambda, \lambda' \in F_{P^\vee} \subset \overline{\mathcal{A}_0}$ such that $\mu := \lambda - \lambda' \in P^\vee$. We have to show that $\mu = 0$. First observe that $-1 \leq \langle \mu, \alpha \rangle \leq 1$ for any root $\alpha \in \Phi$.

Suppose that $\mu \notin \overline{\mathcal{A}_0}$. Since $\langle \mu, \alpha_0 \rangle \leq 1$, there must be some i such that $-1 \leq \langle \mu, \alpha_i \rangle < 0$. But $\langle \mu, \alpha_i \rangle \in \mathbb{Z}$ since we have assumed that $\mu \in P^\vee$, so we get $\langle \mu, \alpha_i \rangle = -1$. Thus, we compute $1 \geq \langle \lambda', \alpha_i \rangle = \langle \lambda, \alpha_i \rangle + 1 \geq 1$ and hence, $\langle \lambda', \alpha_i \rangle = 1$. Furthermore, we have

$$1 \geq \langle \lambda', \alpha_0 \rangle = \sum_j n_j \langle \lambda', \alpha_j \rangle = n_i + \sum_{j \neq i} n_j \underbrace{\langle \lambda', \alpha_j \rangle}_{\geq 0} \geq n_i \geq 1 \;\Rightarrow\; \langle \lambda', \alpha_0 \rangle = n_i = 1$$

and thus, $\langle \lambda', \alpha_0 + \alpha_i \rangle = 2$, a contradiction since $\lambda' \in F_{P^\vee}$.

We have shown that $\mu \in \overline{\mathcal{A}_0} \cap P^\vee$. By [Bou02, VI, §2.2, Prop. 5], if $\mu \neq 0$, then there is some $k \in J$ such that $\mu = \varpi_k^\vee$. Since $\lambda \in F_{P^\vee}$, we have $1 \geq \langle \lambda, \alpha_0 + \alpha_k \rangle = \underbrace{\langle \lambda', \alpha_0 + \alpha_k \rangle}_{\geq 0} + 2 \geq 2$, which is absurd. $\qquad\square$

Type	Extended Dynkin diagram
$\widetilde{A_1}$	
$\widetilde{A_n}$ $(n \geq 2)$	
$\widetilde{B_2} = \widetilde{C_2}$	
$\widetilde{B_n}$ $(n \geq 3)$	
$\widetilde{C_n}$ $(n \geq 3)$	
$\widetilde{D_n}$ $(n \geq 4)$	
$\widetilde{E_6}$	
$\widetilde{E_7}$	
$\widetilde{E_8}$	
$\widetilde{F_4}$	
$\widetilde{G_2}$	

TABLE 2. Extended Dynkin diagrams of irreducible root systems.
The white dots stand for the roots corresponding to minuscule weights and the crossed dots represent the lowest root $\widetilde{\alpha} := -\alpha_0$.

Corollary 2.2.11. *If K is an adjoint group, then $\exp(F_{P^\vee})$ is homeomorphic to the polytope $F_{P^\vee}$ and is a fundamental domain for the action of W on T. In particular, the decomposition*

$$T = \coprod_{w \in W} w \cdot \exp(F_{P^\vee})$$

is cellular, regular and W-equivariant.

The Lemmae 2.2.5 and 2.2.6 may be used to give a useful characterization of the element $w_i = w_0^i w_0$ from the Proposition 2.2.1.

Proposition 2.2.12. *For any $i \in J$, the element w_i is the unique element $w \in W$ of minimal length such that*

$$\varpi_i^\vee \in w w_0(\overline{\mathcal{C}_0}),$$

where w_0 is the longest element of W.

Proof. Let $w \in W$ such that $w^{-1}(\varpi_i^\vee) \in w_0(\overline{\mathcal{C}_0}) = -\overline{\mathcal{C}_0}$ with $\ell(w)$ minimal with respect to this property. First, we prove that $t_{\varpi_i^\vee} w \in \Omega$, that is, $w(\overline{\mathcal{A}_0}) + \varpi_i^\vee = \overline{\mathcal{A}_0}$.

Since $\overline{\mathcal{A}_0} = \mathrm{conv}(\{0\} \cup \{\varpi_j^\vee / n_j\}_{j \in I})$, it is sufficient to prove that for any $j \in I$, we have $w\left(\frac{\varpi_j^\vee}{n_j}\right) + \varpi_i^\vee \in \overline{\mathcal{A}_0}$. Recall that W acts on the set $\{\lambda \in V^* \; ; \; \forall \alpha \in \Phi, \; -1 \leq \langle \lambda, \alpha \rangle \leq 1\}$. This, together with the fact that $-w^{-1}(\varpi_i^\vee) \in \overline{\mathcal{C}_0}$, implies that $-w^{-1}(\varpi_i^\vee) \in \overline{\mathcal{A}_0}$. Thus, we have $-w^{-1}(\varpi_i^\vee) \in \overline{\mathcal{A}_0} \cap P^\vee$ and so there is some $k \in J$ such that $w^{-1}(\varpi_i^\vee) = -\varpi_k^\vee$ (see [Bou02, VI, §2.2, Proposition 5]).

  Claim 1 : For $j \neq i$, we have $w(\varpi_j^\vee) \neq \varpi_j^\vee$.
Suppose the contrary, then

$$w^{-1}\left(\frac{\varpi_j^\vee}{n_j} + \varpi_i^\vee\right) = \frac{\varpi_j^\vee}{n_j} - \varpi_k^\vee$$

and by the Lemma 2.1.2, there exists some $v \in W$ such that $vw^{-1}\left(\frac{\varpi_j^\vee}{n_j} + \varpi_i^\vee\right) \in \overline{\mathcal{A}_0}$ and thus

$$1 \geq \left\langle vw^{-1}\left(\frac{\varpi_j^\vee}{n_j} + \varpi_i^\vee\right), \underbrace{vw^{-1}(\alpha_0)}_{\in \Phi} \right\rangle = \left\langle \frac{\varpi_j^\vee}{n_j} + \varpi_i^\vee, \alpha_0 \right\rangle = 2,$$

a contradiction.

  Claim 2 : For $j \neq i$, we have $\left\langle \varpi_j^\vee, w(\alpha_k) \right\rangle \neq 0$.
Indeed, for each $k' \neq k$, we have $(w s_{k'})^{-1}(\varpi_i^\vee) = -s_{k'}(\varpi_k^\vee) = -\varpi_k^\vee \in -\overline{\mathcal{C}_0}$ and because $\ell(w)$ is minimal, this implies that $\ell(w s_{k'}) = \ell(w) + 1$ and thus $w(\alpha_{k'}) \in \Phi^+$. This yields $\left\langle w^{-1}(\varpi_j^\vee), \alpha_{k'} \right\rangle = \left\langle \varpi_j^\vee, w(\alpha_{k'}) \right\rangle \geq 0$ for each $k' \neq k$. If $\left\langle \varpi_j^\vee, w(\alpha_k) \right\rangle = 0$, then $w^{-1}(\varpi_j^\vee) \in \overline{\mathcal{C}_0}$ and so $w^{-1}\left(\frac{\varpi_j^\vee}{n_j}\right) \in \overline{\mathcal{A}_0} \cap P^\vee$ and we may choose $k' \neq k$ in J such that $w^{-1}\left(\frac{\varpi_j^\vee}{n_j}\right) = \varpi_{k'}^\vee$. Taking the pairing of this equation against $w^{-1}(\alpha_j)$ yields $1/n_j \in \mathbb{Z}$, i.e. $n_j = 1$ and $w^{-1}(\varpi_j^\vee) = \varpi_{k'}^\vee$. Now, the action of W on $P^\vee/Q^\vee$ being trivial, we have that $\varpi_j^\vee \equiv \varpi_{k'}^\vee \pmod{Q^\vee}$ and because $n_j = n_{k'} = 1$, this implies $k' = j$ and thus $w^{-1}(\varpi_j^\vee) = \varpi_j^\vee$. The Claim 1 forbids this, so our assumption that $\left\langle \varpi_j^\vee, w(\alpha_k) \right\rangle = 0$ was wrong.

  Claim 3 : We have $w(\alpha_k) = -\alpha_0$.
Indeed, we have seen that $w(\alpha_{k'}) \in \Phi^+$ for $k' \neq k$ and because $w \neq 1$, this yields $w(\alpha_k) \in \Phi^-$. Now, for $j \neq i$, we have

$$0 \geq \left\langle \varpi_j^\vee, w(\alpha_k) \right\rangle = n_j \left\langle w^{-1}\left(\frac{\varpi_j^\vee}{n_j}\right), \alpha_k \right\rangle$$

and $\left\langle w^{-1}(\varpi_j^\vee), \alpha_k \right\rangle \neq 0$ by the Claim 2, so $-1 \leq \left\langle w^{-1}\left(\frac{\varpi_j^\vee}{n_j}\right), \alpha_k \right\rangle < 0$ and it is an integer, so it equals -1 and thus

$$\forall j \neq i, \; \left\langle \varpi_j^\vee, w(\alpha_k) \right\rangle = -n_j$$

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and we already have $\langle \varpi_i, w(\alpha_k) \rangle = \langle w^{-1}(\varpi_i^\vee), \alpha_k \rangle = -1 = -n_i$. This gives that $w(\alpha_k) = -\alpha_0$, as claimed.

Let $j \in I$. For $j' \neq i$, we compute

$$\left\langle w\left(\frac{\varpi_j^\vee}{n_j}\right) + \varpi_i^\vee, \alpha_{j'} \right\rangle = \left\langle w\left(\frac{\varpi_j^\vee}{n_j}\right), \alpha_{j'} \right\rangle$$

and this is non-negative because $w^{-1}(\alpha_{j'}) \in \Phi^+$. This holds because we must have $\ell(w^{-1}s_{j'}) = \ell(s_{j'}w) = \ell(w) + 1$ since $(s_{j'}w)^{-1}(\varpi_i^\vee) = w^{-1}(\varpi_i^\vee) \in -\overline{\mathcal{C}_0}$ and $\ell(w)$ is minimal with respect to this property. On the other hand, we have

$$\left\langle w\left(\frac{\varpi_j^\vee}{n_j}\right) + \varpi_i^\vee, \alpha_i \right\rangle = 1 + \underbrace{\left\langle w\left(\frac{\varpi_j^\vee}{n_j}\right), \alpha_i \right\rangle}_{\geq -1} \geq 0.$$

This proves that

$$w\left(\frac{\varpi_j^\vee}{n_j}\right) + \varpi_i^\vee \in \overline{\mathcal{C}_0}.$$

Moreover, using the Claim 3 we get

$$\left\langle w\left(\frac{\varpi_j^\vee}{n_j}\right) + \varpi_i^\vee, \alpha_0 \right\rangle = 1 + \left\langle \frac{\varpi_j^\vee}{n_j}, w^{-1}(\alpha_0) \right\rangle = 1 - \left\langle \frac{\varpi_j^\vee}{n_j}, \alpha_k \right\rangle \leq 1$$

and thus $w\left(\frac{\varpi_j^\vee}{n_j}\right) + \varpi_i^\vee \in \overline{\mathcal{A}_0}$, and $\mathrm{t}_{\varpi_i^\vee} w \in \Omega$ as required.

Now, since $\mathrm{t}_{\varpi_i^\vee} w \in \Omega$, the Lemmae 2.2.5 and 2.2.6 are valid for w and let $v := ww_i^{-1}$ where $w_i = w_0^i w_0$ is the element from the Proposition 2.2.1. By these Lemmae, we have $v(\alpha_i) = -w(\alpha_0) = \alpha_i$ and, if $j \neq i$, we have $v(\alpha_j) \in \Pi$ except in the case where $v(\alpha_j) = -\alpha_0$, so $w_i^{-1}(\alpha_j) = -w^{-1}(\alpha_0) = \alpha_k$ with k such that $-w^{-1}(\varpi_i^\vee) = \varpi_k^\vee = -w_i^{-1}(\varpi_i^\vee)$. In this case, we have $\alpha_j = w_i(\alpha_k) = -\alpha_0$, which is excluded. Hence we get $v(\Pi) \subset \Phi^+$ and $v = 1$. $\square$

Remark 2.2.13. *The above characterization may also be used to generalize the construction of the ω_i's to $P^\vee$. More precisely, for any $\lambda \in P^\vee$, consider the element w_λ with minimal length among those $w \in W$ such that $\lambda \in ww_0(\overline{\mathcal{C}_0})$ and define $u_\lambda := \mathrm{t}_\lambda w_\lambda \in \widehat{W}_a$. Note that for $i \in J$, we have $\omega_i = u_{\varpi_i^\vee} \in \Omega$. The assignment $\lambda \mapsto u_\lambda$ results in a well-defined map*

$$P^\vee \xrightarrow{\ u\ } \widehat{W}_a.$$

We finish this section by giving the vertices of the polytope $F_{P^\vee}$.

Proposition 2.2.14. *Let $\mathcal{B}_m$ be the set of all isobarycenters of points in $\{0\} \cup \{\varpi_j^\vee\}_{j \in J}$ with a non-zero coefficient with respect to the origin. In other words,*

$$\mathcal{B}_m := \left\{ \frac{1}{|J'| + 1} \sum_{j \in J'} \varpi_j^\vee \ ; \ J' \subseteq J \right\}$$

$$= \{0\} \cup \left\{ \frac{\varpi_{i_1}^\vee + \cdots + \varpi_{i_k}^\vee}{k+1} \ ; \ 1 \leq k \leq |J| \ \text{ and } \ i_j \in J, \ \forall j \right\}.$$

Then the vertices of the polytope $F_{P^\vee}$ are given by

$$\mathrm{vert}(F_{P^\vee}) = \mathcal{B}_m \cup \left\{ \frac{\varpi_i^\vee}{n_i} \right\}_{i \in I \setminus J}.$$

In particular, if $|J| > 1$ (i.e. if Φ is of type $A_{n>1}$, D_n or E_6), then $|\mathrm{vert}(F_{P^\vee})| > r + 1$, so the cellular decomposition from the Corollary 2.2.11 is no longer simplicial.

Proof. Denote $\mathcal{V} := \mathcal{B}_m \cup \{\varpi_i^\vee/n_i\}_{i\in I\setminus J}$. If $J = \emptyset$, then $\mathcal{B}_m = \{0\}$ and the statement is just the last corollary from [Bou02, VI, §2.2]. So we may assume that J is non-empty. Denote

$$\forall i \in I, \ H_i := \{\lambda \in V^* \ ; \ \langle\lambda, \alpha_i\rangle = 0\} \ \text{ and } \ H_0 := \{\lambda \in V^* \ ; \ \langle\lambda, \alpha_0\rangle = 1\}.$$

We also introduce the following affine hyperplanes

$$\forall j \in J, \ H_j^0 := \{\lambda \in V^* \ ; \ \langle\lambda, \alpha_0 + \alpha_j\rangle = 1\}.$$

By construction, any facet of $F_{P^\vee}$ is of the form $F_{P^\vee} \cap H_i$ for some $0 \leq i \leq r$ or $F_{P^\vee} \cap H_j^0$ for some $j \in J$. Therefore, any face f of $F_{P^\vee}$ is of the form

$$f = F_{P^\vee} \cap \left(\bigcap_{i\in I_f} H_i \cap \bigcap_{j\in J_f} H_j^0 \right)$$

for some subsets $I_f \subseteq I$ and $J_f \subseteq J$ such that $|I_f| + |J_f| = \operatorname{codim}_{F_{P^\vee}}(f) = r - \dim(f)$. In particular

$$\forall v \in \operatorname{vert}(F_{P^\vee}), \ \exists I_v \subseteq I, \ \exists J_v \subseteq J \ ; \ |I_v| + |J_v| = r \ \text{ and } \ \{v\} = F_{P^\vee} \cap \left(\bigcap_{i\in I_v} H_i \cap \bigcap_{j\in J_v} H_j^0 \right).$$

It is straightforward to check that any point of $\mathcal{V}$ is in at least r hyperplanes among $\{H_i, H_j^0 \ ; \ i \in I, \ j \in J\}$, so $\mathcal{V} \subseteq \operatorname{vert}(F_{P^\vee})$.

Conversely, let $v \in \operatorname{vert}(F_{P^\vee})$ and take I_v and J_v as above.

Assume first that $I_v \cap J_v \neq \emptyset$ and let $k \in I_v \cap J_v$. As $\langle v, \alpha_0 + \alpha_k\rangle = 1$ and $\langle v, \alpha_k\rangle = 0$, we get $\langle v, \alpha_0\rangle = 1$. Since $v \in F_{P^\vee}$, this implies that $\langle v, \alpha_j\rangle = 0$ for any $j \in J$, so $J_v = J$. If $J = I$, then $v = 0 \in \mathcal{V}$. Otherwise, there is some $\ell \in I \setminus J$ with $\langle v, \alpha_\ell\rangle > 0$, so $\ell \notin I_v$ and we have

$$\left\{\frac{\varpi_\ell^\vee}{n_\ell}\right\} = F_{P^\vee} \cap \left(\bigcap_{\ell\neq i\in I} H_i \cap \bigcap_{j\in J} H_j^0 \right) \subseteq F_{P^\vee} \cap \left(\bigcap_{i\in I_v} H_i \cap \bigcap_{j\in J_v} H_j^0 \right) = \{v\},$$

thus $v = \varpi_\ell^\vee/n_\ell \in \mathcal{V}$. So if $I_v \cap J_v$ is non-empty, then $v \in \mathcal{V}$.

Now, if $I_v \cap J_v = \emptyset$ we have

$$\left\{\frac{1}{|J_v|+1} \sum_{j\in J_v} \varpi_j^\vee\right\} \subseteq F_{P^\vee} \cap \left(\bigcap_{i\in I_v} H_i \cap \bigcap_{j\in J_v} H_j^0 \right) = \{v\}$$

and thus $v = \frac{1}{|J_v|+1} \sum_{j\in J_v} \varpi_j^\vee \in \mathcal{V}$. $\qquad\square$

Example 2.2.15. *The following figures display the fundamental domain $F_{P^\vee}$ inside the fundamental alcove, itself inside the fundamental chamber for the respective root data of type A_2, B_2 and B_3.*

2.3. The general case.

We now have to see what happens if the W-lattice Y is such that $Q^\vee \subsetneq Y \subsetneq P^\vee$. To simplify notations of this section, we identify a lattice $L \subset V^*$ with its translation group $\operatorname{t}(L) \subset \operatorname{Aff}(V^*)$. We shall give a fundamental domain for the *intermediate affine Weyl group* $W_Y := \operatorname{t}(Y) \rtimes W$ and prove that the quotient map is injective on it. It will be described as a *polytopal complex*. In this paper, by a *polytopal complex* we mean a finite union of closed convex polytopes, each one of which intersecting any other one along a common facet. For a more general definition and treatment of polytopal complexes, we refer the reader to [Zie95, Chapter 5, §5.1].

First, we shall identify Y with a subgroup of Ω. In fact, there is a correspondence between W-lattices $Q^\vee \subseteq \Lambda \subseteq P^\vee$ and the subgroups of Ω. In order to state this correspondence properly, we temporarily drop the letter Y and we work in the root system Φ only.
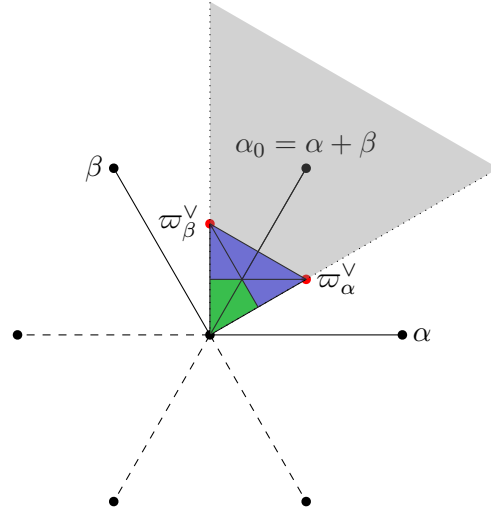
FIGURE 1. The fundamental domain $F_{P^\vee}$ (in green) inside $\overline{\mathcal{A}_0}$ (in blue) inside the Weyl chamber $\overline{\mathcal{C}_0}$ (in gray) in type A_2.
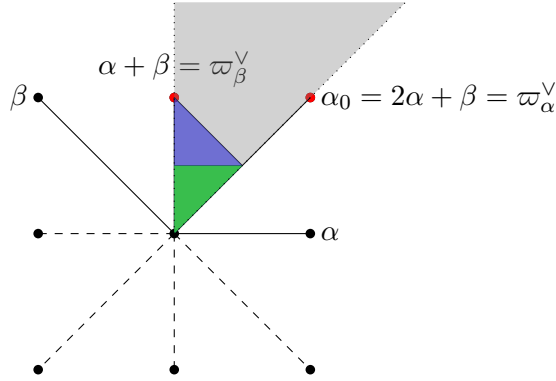

FIGURE 2. The fundamental domain $F_{P^\vee}$ (in green) inside $\overline{\mathcal{A}_0}$ (in blue) inside the Weyl chamber $\overline{\mathcal{C}_0}$ (in gray) in type $B_2 = C_2$.

Proposition 2.3.1. *Recall that $\widehat{W}_{\mathrm{a}} \simeq W_{\mathrm{a}} \rtimes \Omega$ and denote by*

$$\pi : \widehat{W}_{\mathrm{a}} \twoheadrightarrow \Omega$$

the natural projection. Given a W-lattice $Q^\vee \subset \Lambda \subset P^\vee$, we define a subgroup $W_\Lambda := \Lambda \rtimes W \leq \widehat{W}_{\mathrm{a}}$. Then we have a bijective correspondence

$$
\begin{array}{ccc}
\{\Lambda \; ; \; Q^\vee \subseteq \Lambda \subseteq P^\vee \text{ is a W-lattice}\} & \overset{1\text{-}1}{\longleftrightarrow} & \{H \leq \Omega\} \\
\Lambda & \longmapsto & \Omega_\Lambda := \pi(W_\Lambda) \\
\pi^{-1}(H) \cap P^\vee =: \Lambda(H) & \longleftarrow\!\shortmid & H
\end{array}
$$

Moreover, for a W-lattice $Q^\vee \subset \Lambda \subset P^\vee$, we have

$$[\Omega : \Omega_\Lambda] = [P^\vee : \Lambda],$$

or, equivalently

$$|\Omega_\Lambda| = [\Lambda : Q^\vee].$$

Finally, we have a decomposition

$$W_\Lambda \simeq W_{\mathrm{a}} \rtimes \Omega_\Lambda.$$

Proof. First, we prove that the maps $\Omega_\bullet$ and $\Lambda(\bullet)$ are well-defined. It is clear that, given a W-lattice Λ, the set $\Omega_\Lambda = \pi(\Lambda \rtimes W) = \{\phi \in W_\Lambda \; ; \; \phi(\mathcal{A}_0) = \mathcal{A}_0\}$ is a subgroup of Ω. Conversely, le $H \leq \Omega$ be a subgroup. We have that $\Lambda(H) = \pi^{-1}(H) \cap P^\vee$ is a subgroup of $\widehat{W}_{\mathrm{a}}$
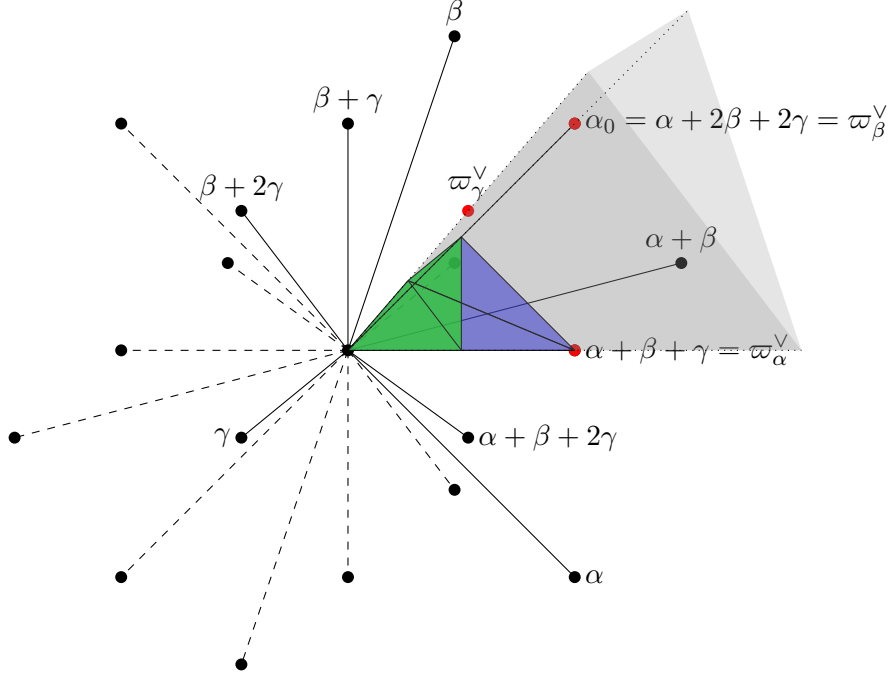
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FIGURE 3. The fundamental domain $F_{P^\vee}$ (in green) inside the fundamental alcove $\overline{\mathcal{A}_0}$ (in blue) in type B_3.

and since $\Lambda(H) \subset P^\vee$, $\Lambda(H)$ is countable, hence discrete. We readily have $Q^\vee \subset \Lambda(H) \subset P^\vee$, which gives $\mathrm{rk}\,(\Lambda(H)) = \mathrm{rk}\,(Q) = \mathrm{rk}\,(\Phi)$. Moreover, if $x \in \Lambda(H)$ and $w \in W$, then we have $\pi(x) = \pi(x \cdot t_0 w) = \pi(t_0 w \cdot x)$ (recall that Ω is abelian) so $wx \in \pi^{-1}(H)$ that is, $wx \in \Lambda(H)$ and $\Lambda(H)$ is indeed a W-lattice lying between $Q^\vee$ and $P^\vee$.

We have to prove that $\Lambda \circ \Omega_\bullet = id$ and $\Omega_{\Lambda(\bullet)} = id$. Take a W-lattice Λ between $Q^\vee$ and $P^\vee$ and let us show that $\Lambda(\Omega_\Lambda) = \Lambda$. If $\mu \in \Lambda$, then $\mu \in P^\vee$ and by construction, $\pi(t_\mu) \in \pi(\Lambda \rtimes W)$ so $\mu \in \pi^{-1}(\pi(W_\Lambda)) \cap P^\vee \overset{\text{def}}{=} \Lambda(\Omega_\Lambda)$. On the other hand, if $\mu \in \pi^{-1}(\pi(W_\Lambda)) \cap P^\vee$, then there are $\nu \in \Lambda$ and $w \in W$ such that $\pi(t_\mu) = \pi(t_\nu w)$, that is, $1 = \pi(t_\nu w t_{-\mu}) = \pi(t_{\nu - w(\mu)} w)$. Hence $t_{\nu - w(\mu)} w \in W_a$, which means that $\nu - w(\mu) \in Q^\vee \subset \Lambda$, but since $\nu \in \Lambda$ and Λ is a W-lattice, we have $\mu \in \Lambda$.

Now, let $H \leq \Omega$ be a subgroup and let us prove that $\Omega_{\Lambda(H)} = H$. Let $x \in \Omega_{\Lambda(H)} = \pi(W_{\Lambda(H)})$. There exist $t_\mu w \in \Lambda(H) \rtimes W$ such that

$$x = \pi(t_\mu w) = \pi(t_\mu 1 \cdot t_0 w) = \pi(t_\mu) \underbrace{\overset{\overset{W_a}{\cup}}{\pi(t_0 w)}}_{=1} = \pi(t_\mu)$$

and this is in H because $\mu \in \Lambda(H) = \pi^{-1}(H) \cap P^\vee$, so $x \in H$. Now, if $y \in H$, then we have $y = \pi(t_\mu w)$ for some $\mu \in P^\vee$ and $w \in W$ because π is onto, but the same calculation as above shows that $y = \pi(t_\mu w) = \pi(t_\mu)$, so $\mu \in P^\vee \cap \pi^{-1}(H) = \Lambda(H)$ and thus $y \in \pi(W_{\Lambda(H)}) = \Omega_{\Lambda(H)}$, as required.

Next, the fact that $[P^\vee : \Lambda] \cdot [\Lambda : Q^\vee] = [P^\vee : Q^\vee] = |\Omega|$ ensures that the two equalities are indeed equivalent. But since $\ker(\pi) = W_a \leq W_\Lambda$ we have

$$\Omega_\Lambda = \pi(W_\Lambda) = \mathrm{im}\,(W_\Lambda \overset{\pi}{\to} \Omega) \simeq W_\Lambda / \ker(\pi_{|W_\Lambda}) = W_\Lambda / \ker(\pi) = W_\Lambda / W_a \simeq \Lambda / Q^\vee.$$

Finally, using the above equality we see that the short exact sequence

$$1 \longrightarrow W_a \longrightarrow W_\Lambda \longrightarrow \Omega_\Lambda \longrightarrow 1$$

is exact. It splits since the short exact sequence

$$1 \longrightarrow W_{\mathrm{a}} \longrightarrow \widehat{W_{\mathrm{a}}} \overset{\pi}{\longrightarrow} \Omega \longrightarrow 1$$

is split. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We have a useful characterization of elements of Ω_Λ, once Λ is known.

Lemma 2.3.2. *Let Λ be a W-lattice between $Q^\vee$ and $P^\vee$ and consider $\omega_i, \omega_j \in \Omega$. Then, the following holds*

  a) *We have $\omega_i \in \Omega_\Lambda$ if and only if $\varpi_i^\vee \in \Lambda$.*
  b) *The group Ω_Λ acts on the set $\overline{\mathcal{A}_0} \cap \Lambda$.*
  c) *We have $\omega_i \Omega_\Lambda = \omega_j \Omega_\Lambda$ if and only if $\varpi_i^\vee - \varpi_j^\vee \in \Lambda$.*

Proof. a) This is obvious, we have

$$\omega_i \in \Omega_\Lambda \iff \omega_i \in W_\Lambda \iff \varpi_i^\vee = \omega_i(0) \in \Lambda.$$

  b) If $\lambda \in \overline{\mathcal{A}_0} \cap \Lambda$, then in particular we have $\lambda \in \overline{\mathcal{A}_0} \cap P^\vee$ so by [Bou02, VI, §2.2, Proposition 5] there is some $k \in J$ such that $\lambda = \varpi_k^\vee \in \Lambda$. Thus, if $\omega_i \in \Omega_\Lambda$, we have $\omega_i(\lambda) = \omega_i(\varpi_k^\vee) = w_i(\varpi_k^\vee) + \varpi_i^\vee \in \Lambda$ because $\varpi_i^\vee \in \Lambda$ by $a)$ and $w_i(\varpi_k^\vee) \in \Lambda$ since Λ is a W-lattice.

  c) First, we compute

$$\omega_j^{-1}\omega_i = (\mathrm{t}_{\varpi_j^\vee} w_j)^{-1} \mathrm{t}_{\varpi_i^\vee} w_i = w_j^{-1} \mathrm{t}_{\varpi_i^\vee - \varpi_j^\vee} w_i = \mathrm{t}_{w_j^{-1}(\varpi_i^\vee - \varpi_j^\vee)} w_j^{-1} w_i.$$

Thus, we have

$$\omega_j^{-1}\omega_i \in \Omega_\Lambda \iff \mathrm{t}_{w_j^{-1}(\varpi_i^\vee - \varpi_j^\vee)} w_j^{-1} w_i \in \Omega_\Lambda \iff w_j^{-1}(\varpi_i^\vee - \varpi_j^\vee) \in \Lambda \iff \varpi_i^\vee - \varpi_j^\vee \in \Lambda.$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$$

Return to the case $\Lambda = Y$ and recall the notations from the preceding subsection: $I = \{1, \dots, r\}$, $\Pi = \{\alpha_i, \ i \in I\}$, $\alpha_0 = \sum_i n_i \alpha_i$ and $J = \{i \in I \ ; \ n_i = 1\}$. Recall also that the polytope

$$F := F_{P^\vee} = \{\lambda \in \overline{\mathcal{A}_0} \ ; \ \forall j \in J, \ \langle \lambda, \alpha_0 + \alpha_i \rangle \leq 1\}$$

given in Proposition 2.2.3 is a fundamental domain for Ω in $\overline{\mathcal{A}_0}$.

Proposition 2.3.3. *The following holds*

$$\forall \omega \in \Omega \setminus \{1\}, \ \mathrm{codim}_F(F \cap \omega F) = 1.$$

More precisely, for $1 \neq \omega \in \Omega$, the polytopes F and ωF intersect along their common facet $F \cap \{\lambda \in V^* \ ; \ \langle \lambda, \alpha_0 + \alpha_j \rangle = 1\}$, where $j \in J$ is such that $\omega(\varpi_j^\vee) = 0$.

Proof. We can write $\omega = \omega_i$ for some $i \in J$. Observe that if $\omega(\varpi_j^\vee) = 0$ then, using the Lemmae 2.2.5 and 2.2.6, we compute for $\lambda \in V^*$,

$$\left\langle \omega^{-1}(\lambda), \alpha_0 + \alpha_j \right\rangle = \left\langle w_i^{-1}\lambda + \varpi_j^\vee, \alpha_0 + \alpha_j \right\rangle = 2 + \langle \lambda, w_i\alpha_0 + w_i\alpha_j \rangle = 2 - \langle \lambda, \alpha_0 + \alpha_i \rangle.$$

Hence, we have

(2) $$\qquad\qquad \omega_i(\varpi_j^\vee) = 0 \ \Rightarrow \ \forall \lambda \in V^*, \ \langle \lambda, \alpha_0 + \alpha_j \rangle + \langle \omega_i(\lambda), \alpha_0 + \alpha_i \rangle = 2.$$

This being said, let us prove that

$$F \cap \{\lambda \ ; \ \langle \lambda, \alpha_0 + \alpha_i \rangle = 1\} = F \cap \omega F.$$

Let $\lambda \in F$ such that $\langle \lambda, \alpha_0 + \alpha_i \rangle = 1$. By (2), we have $\left\langle \omega^{-1}(\lambda), \alpha_0 + \alpha_j \right\rangle = 1$ and if $k \neq j$, we have $w_i\alpha_k = \alpha_{k'}$ for some $k' \in J$ and we compute

$$\left\langle \omega^{-1}(\lambda), \alpha_0 + \alpha_k \right\rangle = \left\langle w_i^{-1}\lambda + \varpi_j^\vee, \alpha_0 + \alpha_k \right\rangle = 1 + \langle \lambda, \alpha_{k'} - \alpha_i \rangle = \langle \lambda, \alpha_0 + \alpha_{k'} \rangle \leq 1.$$

Hence $\omega^{-1}(\lambda) \in F$ and so $\lambda \in F \cap \omega F$.

Conversely, if $\lambda \in F \cap \omega F$, then by (2) we get

$$1 \geq \langle \lambda, \alpha_0 + \alpha_i \rangle = 2 - \underbrace{\langle \omega^{-1}(\lambda), \alpha_0 + \alpha_j \rangle}_{\leq 1} \geq 1$$

hence $\langle \lambda, \alpha_0 + \alpha_i \rangle = 1$, as required. $\qquad\square$

Corollary 2.3.4. *To the W-lattice Y we associate the following subgroup Ω_Y of Ω from Proposition 2.3.1:*

$$\Omega_Y = \{\phi \in W_Y \ ; \ \phi(\mathcal{A}_0) = \mathcal{A}_0\} \leq \Omega.$$

Choose a representative $\omega \in \Omega$ for each coset $[\omega] \in \Omega/\Omega_Y$ and define

$$F_Y := \bigcup_{[\omega] \in \Omega/\Omega_Y} \omega \cdot F_{P^\vee}.$$

Then F_Y is a polytopal complex and is a fundamental domain for the action of Ω_Y on $\overline{\mathcal{A}_0}$.

Proof. By the Proposition 2.3.3, for every subset $A \subseteq \Omega$, the subset $\bigcup_{a \in A} aF$ is a connected closed polytopal complex.

The rest of this proof is very standard. Let $k := [P^\vee : Y] = [\Omega : \Omega_Y]$, choose representatives $x_1, \ldots, x_k \in \Omega$ such that $\Omega/\Omega_Y = \{x_i \Omega_Y, \ 1 \leq i \leq k\}$ and define $F_Y := \bigcup_{i=1}^k x_i F_{P^\vee}$. Let $\lambda \in \overline{\mathcal{A}_0}$. Since $F_{P^\vee}$ is a fundamental domain for $\widehat{W}_{\mathrm{a}}$, there is some $\omega \in \Omega$ such that $\omega^{-1}(\lambda) \in F_{P^\vee}$ and writing $\omega = u x_i$ for some $1 \leq i \leq k$ and some $u \in \Omega_Y$, we obtain $u^{-1}\lambda \in x_i F_{P^\vee} \subset F_Y$. On the other hand, if $u \in \Omega_Y$, then

$$F_Y \cap u F_Y = \bigcup_{1 \leq i,j \leq k} x_i F_{P^\vee} \cap u x_j F_{P^\vee} = \bigcup_{i,j} x_i (F_{P^\vee} \cap x_i^{-1} u x_j F_{P^\vee}).$$

If $F_Y \cap u F_Y$ has non-empty interior, then there are some i, j such that $F_{P^\vee} \cap u x_j x_i^{-1} F_{P^\vee}$ has non-empty interior. This implies that $u x_j x_i^{-1} = 1$, thus $x_i x_j^{-1} \in \Omega_Y$ so $i = j$ and $u = 1$. $\quad\square$

From this, we deduce the main result of this section:

Theorem 2.3.5. *If K is only supposed to be compact, let $W_Y := \mathrm{t}(Y) \rtimes W$ (where $Y = Y(T)$ is the cocharacter lattice) and consider the subgroup*

$$\Omega_Y = \{\phi \in W_Y \ ; \ \phi(\mathcal{A}_0) = \mathcal{A}_0\} \leq \Omega$$

and choose a representative for each coset in Ω/Ω_Y. Define

$$F_Y := \bigcup_{[\omega] \in \Omega/\Omega_Y} \omega \cdot F_{P^\vee},$$

where $F_{P^\vee}$ is the fundamental domain for Ω in $\overline{\mathcal{A}_0}$ given in Proposition 2.2.3. Then F_Y is a polytopal complex and is a fundamental domain for the action of $W_Y \simeq W_{\mathrm{a}} \rtimes \Omega_Y$ on V^*.

We still have to investigate the question of the injectivity of the exponential map on F_Y. This is done in the following proposition:

Proposition 2.3.6. *The hypothesis of the Lemma 1.2.1 is fulfilled under the ones of the Theorem 2.3.5. In other words, one has*

$$F_Y \cap (Y \setminus \{0\} + F_Y) = \emptyset.$$

Proof. Let $\lambda, \lambda' \in F_Y$ and suppose by contradiction that $0 \neq \mu := \lambda - \lambda' \in Y$. In particular we have $\mu \in P^\vee$ and since $\lambda, \lambda' \in \overline{\mathcal{A}_0}$, we have $\langle \mu, \alpha \rangle \in \{-1, 0, 1\}$ for every $\alpha \in \Phi$. Without loss of generality, as Ω_Y acts on Y, we may assume that the trivial coset in Ω/Ω_Y is represented by $1 \in \Omega$.

First, suppose that $\mu \notin \overline{\mathcal{A}_0}$. Since $\langle \mu, \alpha_0 \rangle \leq 1$, we may choose $i \in I$ such that $\langle \mu, \alpha_i \rangle = -1$. Then, we have $1 \geq \langle \lambda', \alpha_i \rangle = \langle \lambda, \alpha_i \rangle + 1 \geq 1$ so $\langle \lambda', \alpha_i \rangle = 1$ and thus $\langle \lambda', \alpha_k \rangle = 0$ for $k \neq i$ since $\lambda' \in \overline{\mathcal{A}_0}$ and therefore $\lambda' = \varpi_i^\vee$.

Now, if there is some j such that $\langle \mu, \alpha_j \rangle \neq 0$, then $\langle \mu, \alpha_j \rangle = 1$ and similarly as above, we get $\langle \lambda, \alpha_j \rangle = 1$ and $\lambda = \varpi_j^\vee$. Notice that $j \in J$ because $\lambda \in \overline{\mathcal{A}_0}$.

<u>Claim</u> : For every $h \in J$ and every $\omega \in \Omega$, we have $\omega = \omega_h$ as soon as $\varpi_h^\vee \in \omega F_{P^\vee}$. Indeed, it suffices to show that $\omega^{-1}(\varpi_h^\vee) = 0$. Suppose otherwise, then there exists $\ell \in J$ such that $\omega^{-1}(\varpi_h^\vee) = \varpi_\ell^\vee$ and this would be in $F_{P^\vee}$, a contradiction since $\langle \varpi_\ell^\vee, \alpha_0 + \alpha_\ell \rangle = 2$ and $\ell \in J$.

This proves that if $\lambda = \varpi_j^\vee \in \omega F_{P^\vee}$ for some ω, then $\omega = \omega_j$ and similarly for λ'. Since F_Y is the union of the $\omega F_{P^\vee}$ for ω describing a set of coset representatives modulo Ω_Y, and since $\lambda - \lambda' = \varpi_j^\vee - \varpi_i^\vee \in Y$, we must have $\omega_i \Omega_Y = \omega_j \Omega_Y$ by the Lemma 2.3.2 and thus $\lambda = \lambda'$, a contradiction.

Hence, if $\mu \notin \overline{\mathcal{A}_0}$, then for all $k \neq i$, we have $\langle \mu, \alpha_k \rangle = 0$, so $\mu = -\varpi_i^\vee \in Y$ and $\lambda' = \lambda + \varpi_i^\vee$. Henceforth, $\lambda' = \varpi_i^\vee \in Y$ and the only $\omega \in \Omega$ such that $\varpi_i^\vee \in \omega F_{P^\vee}$ is $\omega = \omega_i$ by the claim and this is in Ω_Y by the Lemma 2.3.2. By definition of F_Y and since the trivial coset in Ω/Ω_Y is represented by 1, this implies that $\varpi_i^\vee \in Y \cap F_{P^\vee} \subset P^\vee \cap F_{P^\vee} = \{0\}$ by the Lemma 2.2.10, a contradiction again.

The only remaining possibility is $\mu \in \overline{\mathcal{A}_0}$. But in this case $\mu \in Y \cap \overline{\mathcal{A}_0}$ and $\mu = \varpi_i^\vee \in Y$ for some $i \in J$. Therefore, we obtain as above $\langle \lambda, \alpha_i \rangle = \langle \lambda', \alpha_i \rangle + 1$, so $\langle \lambda, \alpha_i \rangle = 1$ and so $\lambda = \varpi_i^\vee \in Y$, but we have seen this to be impossible. $\qquad \square$

Corollary 2.3.7. *If K is a compact connected Lie group and if $Y := X(T)^\vee$ is the cocharacter lattice of the chosen maximal torus $T \leq K$, let F_Y be the fundamental domain from the Theorem 2.3.5. Then $\exp(F_Y)$ is homeomorphic to the polytopal complex F_Y and is a fundamental domain for the action of W on T. In particular, the decomposition*

$$T = \coprod_{w \in W} w \cdot \exp(F_Y)$$

is cellular, regular and W-equivariant.

There is another natural choice for the fundamental domain of W_Y in V^*, which is a convex polytope. Unfortunately, the projection map fails to be injective on this domain in general (see Example 2.3.10), making it unsuitable for cell transport to the torus T via exp.

Proposition 2.3.8. *Let*

$$J_Y := \{j \in J \ ; \ \varpi_j^\vee \in Y\} = \{j \in J \ ; \ \omega_j \in \Omega_Y\}$$

and define

$$F_Y' := \{\lambda \in \overline{\mathcal{A}_0} \ ; \ \forall j \in J_Y, \ \langle \lambda, \alpha_0 + \alpha_j \rangle \leq 1\}.$$

Then F_Y' is a convex polytope and is a fundamental domain for $W_Y = \mathrm{t}(Y) \rtimes W$ on V^*.

Proof. We may proceed as in the adjoint case, replacing Ω by Ω_Y. It clearly suffices to show that F_Y' is a fundamental domain for Ω_Y in $\overline{\mathcal{A}_0}$.

First, since F_Y' is an intersection of closed half-spaces, it is a convex polytope in $\overline{\mathcal{A}_0}$. In particular, it is closed and connected.

If $\omega_i \in \Omega_Y$ and $\lambda \in F_Y'$, then there exists $j \in J$ such that $\omega_i(\varpi_j^\vee) = 0$ and we have $j \in J_Y$ since Ω_Y acts on $Y \cap \overline{\mathcal{A}_0}$. We have

$$\langle \omega_i(\lambda), \alpha_0 + \alpha_i \rangle = 2 - \langle \lambda, \alpha_0 + \alpha_j \rangle$$

and

$$F_Y' \cap \omega_i^{-1} F_Y' \subset \overline{\mathcal{A}_0} \cap \{\mu \in V^* \ ; \ \langle \mu, \alpha_0 + \alpha_i \rangle = 1\}$$

and thus $F_Y' \cap \omega_i^{-1} F_Y'$ has empty interior in $\overline{\mathcal{A}_0}$

Let now $\lambda \in \overline{\mathcal{A}_0}$ and choose $i \in J_Y$ such that $\langle \lambda, \alpha_i \rangle = \max_{j \in J_Y} \langle \lambda, \alpha_j \rangle$. If $\langle \lambda, \alpha_0 + \alpha_i \rangle \leq 1$, then $\lambda \in F_Y'$. Otherwise, choose $j \in J_Y$ such that $\omega_i(\varpi_j^\vee) \neq 0$ and take some $k \in J$ such that $\omega_i(\varpi_j^\vee) = \varpi_k^\vee$. Then $k \in J_Y$ because $j \in J_Y$ and $\omega_i \in \Omega_Y$ by the Lemma 2.3.2. Hence

$$\left\langle \omega_i^{-1}(\lambda), \alpha_0 + \alpha_j \right\rangle = 1 + \underbrace{\langle \lambda, \alpha_k - \alpha_i \rangle}_{\leq 0} \leq 1.$$

Now, if $j \in J_Y$ is such that $\omega_i(\varpi_j^\vee) = 0$, then $w_i(\alpha_j) = -\alpha_0$ and

$$\langle \omega_i^{-1}(\lambda), \alpha_0 + \alpha_j \rangle = 2 - \underbrace{\langle \lambda, \alpha_0 + \alpha_i \rangle}_{>1} < 1$$

and thus $\omega_i^{-1}(\lambda) \in F_Y'$, as required. $\qquad\qquad\square$

Reproducing the proof of the Proposition 2.2.14 *verbatim* in this case yields the following description of the vertices of F_Y':

**Proposition 2.3.9.** *Define the following set of isobarycenters*

$$\mathcal{B}_m^Y := \left\{ \frac{1}{|J_Y'| + 1} \sum_{j \in J_Y'} \varpi_j^\vee \; ; \; \emptyset \subseteq J_Y' \subseteq J_Y \right\},$$

where $J_Y = \{j \in J \; ; \; \varpi_j^\vee \in Y\}$. Then the vertices of the polytope F_Y' are given by

$$\mathrm{vert}(F_Y') = \mathcal{B}_m^Y \cup \left\{ \frac{\varpi_i^\vee}{n_i} \right\}_{i \in I \setminus J_Y}.$$

Example 2.3.10. *This fundamental domain F_Y' has the bad taste of containing different points that are congruent modulo Y.*

For instance, let $(X(T), \Phi, Y(T), \Phi^\vee)$ be the root datum of the Lie group $SU_4(\mathbb{C})/\{\pm 1\}$, with maximal torus being the the image in the quotient of the diagonal matrices of SU_4. In particular, Φ is of type A_3. Denote the simple roots by $\Pi := \{\alpha, \beta, \gamma\}$ with extended Dynkin diagram
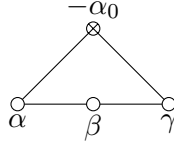
Let also $\varpi_\alpha^\vee$, $\varpi_\beta^\vee$ and $\varpi_\gamma^\vee$ be the corresponding fundamental weights. We have (see [Hum72, §13.2, Table 1])

$$\varpi_\alpha = \frac{3\alpha + 2\beta + \gamma}{4}, \quad \varpi_\beta = \frac{2\alpha + 4\beta + 2\gamma}{4}, \quad \varpi_\gamma = \frac{\alpha + 2\beta + 3\gamma}{4},$$

and every fundamental weight is minuscule since the highest root is $\alpha_0 = \alpha + \beta + \gamma$. In this case, the cocharacter lattice $Y = Y(T)$ is given by

$$Y = \mathbb{Z}\langle \alpha^\vee \rangle \oplus \mathbb{Z}\langle \beta^\vee \rangle \oplus \mathbb{Z}\left\langle \frac{\alpha^\vee + \gamma^\vee}{2} \right\rangle =: \mathbb{Z}\langle \alpha^\vee \rangle \oplus \mathbb{Z}\langle \beta^\vee \rangle \oplus \mathbb{Z}\langle y \rangle.$$

We have

$$\gamma^\vee = 2y - \alpha^\vee \in Y, \quad \varpi_\alpha^\vee = \frac{\alpha^\vee + \beta^\vee + y}{2} \notin Y, \quad \varpi_\gamma^\vee = \frac{\beta^\vee - \alpha^\vee + 3y}{2} \notin Y$$

and hence $Q^\vee \subsetneq Y \subsetneq P^\vee$. Furthermore, $Y \cap \overline{\mathcal{A}_0} = \{\varpi_\beta^\vee\}$ and

$$F_Y' = \{\lambda \in \overline{\mathcal{A}_0} \; ; \; \langle \lambda, \alpha_0 + \beta \rangle \le 1\} = \{\lambda \in \mathrm{conv}(0, \varpi_\alpha^\vee, \varpi_\beta^\vee, \varpi_\gamma^\vee) \; ; \; \langle \lambda, \alpha + 2\beta + \gamma \rangle \le 1\}.$$

So $\varpi_\alpha^\vee$ and $\varpi_\gamma^\vee$ are distinct elements of F_Y' and however,

$$\varpi_\gamma^\vee - \varpi_\alpha^\vee = \varpi_\beta^\vee - \alpha^\vee - \beta^\vee \in Y.$$

Note that, in this case we have

$$\Omega = \langle \omega_\alpha \rangle = \{1, \underbrace{t_{\varpi_\alpha^\vee}(s_\alpha s_\beta s_\gamma)}_{\omega_\alpha}, \underbrace{t_{\varpi_\beta^\vee}(s_\beta s_\gamma s_\alpha s_\beta)}_{\omega_\beta}, \underbrace{t_{\varpi_\gamma^\vee}(s_\gamma s_\beta s_\alpha)}_{\omega_\gamma}\} \simeq \mathbb{Z}/4\mathbb{Z}$$

and

$$\Omega_Y = \langle \omega_\beta \rangle = \{1, \omega_\beta\} \le \Omega \quad and \quad \Omega/\Omega_Y \simeq \pi_1(SU_4(\mathbb{C})/\{\pm 1\}) \simeq \mathbb{Z}/2\mathbb{Z}.$$

By the Proposition 2.3.1, we see that Y is the only W-lattice that stands strictly between $Q^\vee$ and $P^\vee$. Thus, the group $SU_4(\mathbb{C})/\{\pm 1\}$ is the only non-adjoint and non-simply-connected compact Lie group of type A_3.

Note that in this case, we have indeed

$$F_{P^\vee} = \text{conv}\left(0, \frac{\varpi_\alpha}{2}, \frac{\varpi_\beta}{2}, \frac{\varpi_\gamma}{2}, \frac{\varpi_\alpha + \varpi_\beta}{3}, \frac{\varpi_\beta + \varpi_\gamma}{3}, \frac{\varpi_\alpha + \varpi_\gamma}{3}, \frac{\varpi_\alpha + \varpi_\beta + \varpi_\gamma}{4}\right).$$
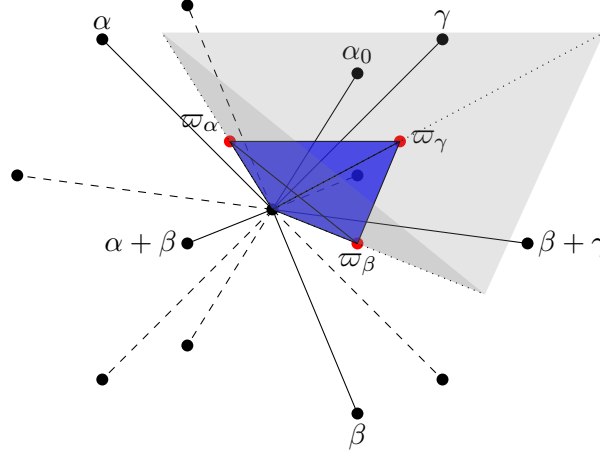

FIGURE 4. The fundamental alcove $\overline{\mathcal{A}_0} = \text{conv}(0, \varpi_\alpha, \varpi_\beta, \varpi_\gamma)$ in type A_3.
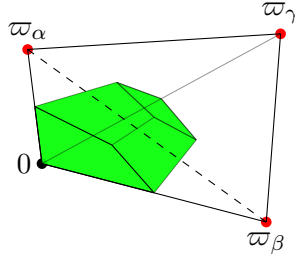

FIGURE 5. The fundamental domain $F_{P^\vee}$ for $\Omega \simeq \mathbb{Z}/4\mathbb{Z}$ in $\overline{\mathcal{A}_0}$.
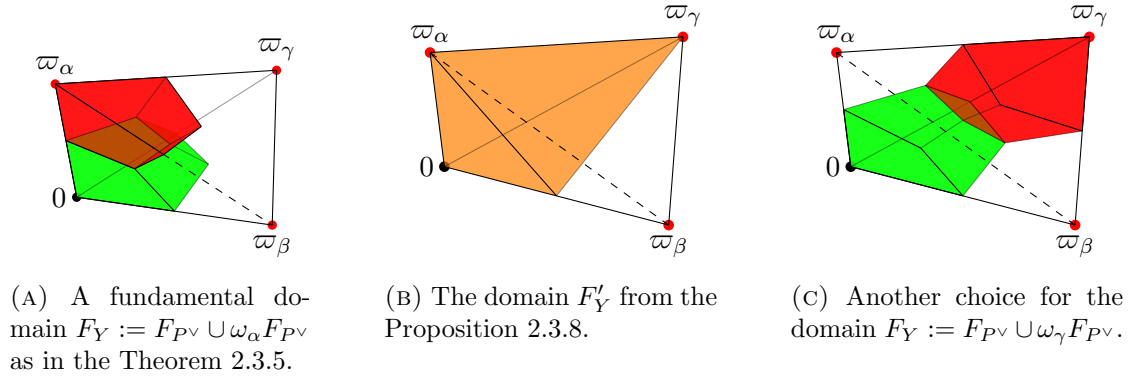

(A) A fundamental domain $F_Y := F_{P^\vee} \cup \omega_\alpha F_{P^\vee}$ as in the Theorem 2.3.5.

(B) The domain F_Y' from the Proposition 2.3.8.

(C) Another choice for the domain $F_Y := F_{P^\vee} \cup \omega_\gamma F_{P^\vee}$.

FIGURE 6. Fundamental domains F_Y and F_Y' inside $\overline{\mathcal{A}_0}$ in type A_3.

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