WEAK IMAGE AND HEMISPHERICAL CATEGORIES

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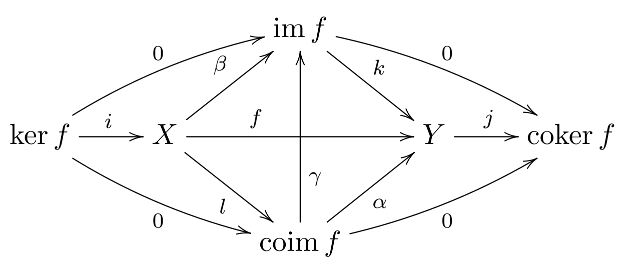
Abstract. The main purpose of this work is to give an alternative definition of the concept of image of a morphism in a general (locally small) category, which seems to be more intuitive than the usual definition, namely the kernel of the cokernel. This new concept allows us to generalize Noether's isomorphism theorem to a new class of categories, which encompasses abelian categories. We will name this class "hemispherical categories". Of course, in an abelian category, weak image equals the kernel of the cokernel. Furthermore, we shall see that the category $\mathfrak{Grp}$ (resp. $\mathfrak{Ann}_0$) of groups (resp. pseudo-rings) is hemispherical, and then we get a formal proof of the isomorphism theorem, valid in $\mathfrak{Grp}$ and in $\mathfrak{Ann}_0$. Finally, we will define the concept of "semi-normal monomorphisms", which extends the one of normal monomorphisms to hemispherical categories.

1. Global Diagram and Weak Image

First of all, the notion that leads to the adjective "hemispherical" is that of global diagram of a morphism, which is not really a formal concept, but a visual way to apprehend a morphism in a category, and its relations to its usual (co)equalizers (see [4], Definition 3.1.10).

Definition 1. *Let $\mathcal{C}$ be a category, X, Y be two objects in $\mathcal{C}$ and $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$. The global diagram of f is the commutative diagram in which one writes f, its kernel, cokernel, image and coimage (if they exist) and their relations.*

Example 1. *If $\mathcal{C}$ is abelian, the global diagram of $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is given by*



where γ is an isomorphism, by the usual isomorphism theorem for abelian categories.

Definition 2. *Let $\mathcal{C}$ be a category and $f : X \to Y$ a morphism in $\mathcal{C}$. An object W, together with morphisms $i : W \to Y$ and $p : X \to W$ is called a weak image of f if the following conditions are satisfied*

(1) $i : W \to Y$ is a monomorphism,
(2) $f = i \circ p$
(3) If K is an object in $\mathcal{C}$, if $j : K \to Y$ is a monomorphism and if $q : X \to K$ is a morphism such that $f = j \circ q$ then there exists a unique $\delta : W \to K$ such that $\delta \circ p = q$ and $j \circ \delta = i$.

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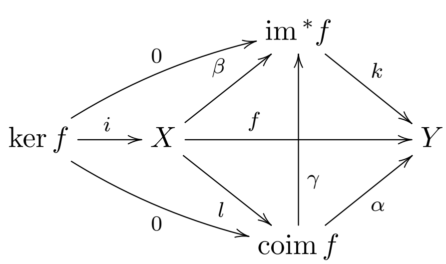
This definition finds its origin in a footnote from section 1.3 of Grothendieck's Tôhoku [2], about the concept of image in an additive category[1]. Note that in some literature (such as [3], 1.13), the image is defined just the way we did here ; and then it is proved ([3], 4.2, lemma 3) that this image is the kernel of the cokernel in an abelian category.

Remark 1. *(1) If a weak image exists, then it is unique up to isomorphism, as one can directly check. The weak image of f (if it exists) will be denoted by $\operatorname{im}^* f$.*
(2) It is not sufficient, a priori, *to suppose $\mathcal{C}$ to be pre-abelian to get an isomorphism $\operatorname{im} \simeq \operatorname{im}^*$. However, this last statement is true if $\mathcal{C}$ is abelian, as we shall see.*

2. Hemispherical categories and generalized Noether isomorphism theorem

Definition 3. *A category $\mathcal{H}$ is said to be hemispherical if the following conditions are verified :*
(1) $\mathcal{H}$ admits a zero object,
(2) Every morphism in $\mathcal{H}$ has a kernel, a coimage and a weak image,
(3) For every morphism $f : X \to Y$, the canonical morphism $\operatorname{coim} f \to Y$ is a monomorphism.

Remark 2. *(1) If $\mathcal{H}$ satisfies only the first and the second axiom above, it is said to be pre-hemispherical.*
(2) The word "hemispherical" comes from the fact that, in such categories, the sketch of the global diagram of $f : X \to Y$ is



Then one may say that abelian categories are "spherical", as seen in our first example. To say it in an explicite (but pedantic !) way, in a hemispherical category, global diagrams are hermispherical.

Example 2. *The categories $\mathfrak{Grp}$ and $\mathfrak{Ann}_0$ are hemispherical, keeping in mind that objects in $\mathfrak{Ann}_0$ are pseudo-rings, i.e. rings without imposing the existence of a neutral element ; in order to get a zero object in $\mathfrak{Ann}_0$. For instance, we check the case of $\mathfrak{Grp}$. The only one condition which is not clear is that every morphism has a weak image. So, let $f : G \to K$ a group homomorphism. Define*

$$W := \{f(g),\ g \in G\},$$

as well as

$$\begin{array}{cccc} p & : & X & \to & W \\ & & g & \mapsto & f(g) \end{array}$$

[1] "Une définition, plus naturelle à vrai dire, de l'image de $u : A \to B$, serait de prendre le plus petit sous-truc B' de B (s'il en existe) tel que u provienne d'un morphisme de A dans B'. Cette définition n'est équivalente à celle donnée dans le texte que dans le cas où $\mathcal{C}$ est une catégorie abélienne (cf. 1.4)."

and

$$i : W \hookrightarrow Y$$

the natural embedding. i *is injective, so it is a monomorphism and one clearly has* $f = i \circ p$. *Let* H *be a group and let* $j : H \to K$ *a monomorphism and* $q : G \to H$ *such that* $f = j \circ q$. *Denote by* $\epsilon : \ker f \hookrightarrow G$ *the natural injection. Since* j *is a monomorphism and* $j \circ q \circ \epsilon = f \circ \epsilon = 1 = j \circ 1$, *we get* $q \circ \epsilon = 1$. *In this case, if* $f(g') = f(g)$, *then* $g'g^{-1} \in \ker f$ *and so,* $q(g'g^{-1}) = q(\epsilon(g'g^{-1})) = 1$ *which implies that* $q(g') = q(g)$. *These considerations show that the map*

$$\begin{array}{cccc} \delta & : & W & \to & H \\ & & f(g) & \mapsto & q(g) \end{array}$$

is well-defined and is a group homomorphism. Furthermore, one has $\delta \circ p(g) = \delta \circ f(g) = q(g)$ *for all* $g \in G$, *as well as* $j \circ \delta(h) = j \circ \delta \circ f(g) = j \circ q(g) = f(g) = i \circ f(g) = i(h)$ *if* $h = f(g) \in W$. *At last, we see that in order to get the equation* $\delta \circ p = q$, *one has to define* δ *as above, so* $W = \operatorname{im}^* f$ *is indeed a weak image for* f.

Lemma 1. *Let* $\mathcal{C}$ *be a category, with a zero object and* $f \in \operatorname{Mor}_{\mathcal{C}}(X, Y)$ *be a morphism.*

(1) If f *has a kernel, then the morphism*

$$\ker f \xrightarrow{\iota} X$$

is a monomorphism.

(2) If f *has a cokernel, then the morphism*

$$Y \xrightarrow{\pi} \operatorname{coker} f$$

is an epimorphism.

Proof. By duality, one only has to deal with the first case. Let A be an object in $\mathcal{C}$ and $\alpha, \beta : A \to \ker f$ such that $\iota \circ \alpha = \iota \circ \beta$. Since $f \circ \iota \circ \alpha = 0 = f \circ \iota$, the universal property of the kernel gives a unique morphism $\gamma : A \to \ker f$ such that $\iota \circ \gamma = \iota \circ \alpha$. In the same way, there is a unique $\delta : A \to \ker f$ such that $\iota \circ \delta = \iota \circ \beta$. But the equations

$$\iota \circ \beta = \iota \circ \alpha,$$

$$\iota \circ \gamma = \iota \circ \beta$$

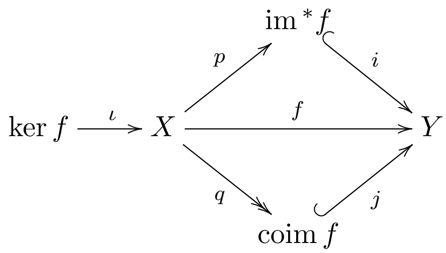
imply that $\gamma = \beta$, $\delta = \alpha$ and $\gamma = \delta$. So $\alpha = \beta$, as we had to show. □

We shall now move on to the main result, namely the generalization of the isomorphism theorem.

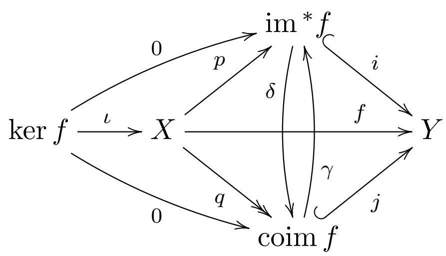
Theorem 1. *Let* $\mathcal{H}$ *be a hemispherical category and* $f : X \to Y$ *a morphism in* $\mathcal{H}$. *Then there exists a unique isomorphism* $\gamma : \operatorname{coim} f \xrightarrow{\sim} \operatorname{im}^* f$ *making the following square commutative*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle i} \\ \operatorname{coim} f & \xrightarrow[\gamma]{\sim} & \operatorname{im}^* f \end{array}$$

Proof. First of all, it follows from Lemma 1 that $q : X \to \operatorname{coim} f = \operatorname{coker}(\ker f)$ is an epimorphism. Writing the commutative diagram

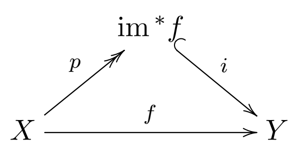


one has $i \circ p \circ \iota = f \circ \iota = 0$ and since i is a monomorphism, one gets $p \circ \iota = 0$. By the universal property of the coimage, there is a unique $\gamma : \operatorname{coim} f \to \operatorname{im}^* f$ such that $\gamma \circ q = p$. By another way, one has $i \circ \gamma \circ q = i \circ p = f = j \circ q$ so $i \circ \gamma = j$ because q is an epimorphism. Since j is a monomorphism, the universal property of the weak image gives a unique morphism $\delta : \operatorname{im}^* f \to \operatorname{coim} f$ such that $\delta \circ p = q$ and $j \circ \delta = i$. Consequently, one gets the following (commutative) diagram



Next, one has $\delta \circ \gamma \circ q = \delta \circ p = q = id_{\operatorname{coim} f} \circ q$ and since q is a epimorphism, we may conclude that $\delta \circ \gamma = id_{\operatorname{coim} f}$. Finally, recalling that i is a monomorphism, since $i \circ \gamma \circ \delta = j \circ \delta = i = i \circ id_{\operatorname{im}^* f}$ one obtains $\gamma \circ \delta = id_{\operatorname{im}^* f}$. Hence, γ is an isomorphism, as desired. □

Corollary 1. *If $f : X \to Y$ is a morphism in a hemispherical category, then f has an epi-monic factorization, through its weak image :*



3. Comparison between hemispherical and abelian categories

After these observations arises a natural question : Are abelian categories hemispherical ? Conveniently, the answer is yes :

Proposition 1. *Every abelian category is hemispherical.*

Proof. Let $\mathcal{A}$ be an abelian category and $f \in \operatorname{Hom}_{\mathcal{A}}(X, Y)$ be a homomorphism. Recalling the proof of the fifth theorem from [1], the morphism $i : \operatorname{coim} f \to Y$ is a monomorphism. Hence, the weak image may be defined by the coimage. Indeed, given K an object in $\mathcal{A}$, $j : K \to Y$ a monomorphism and $q : X \to K$ such that $f = j \circ q$. We have $j \circ q \circ \iota = f \circ \iota = 0$

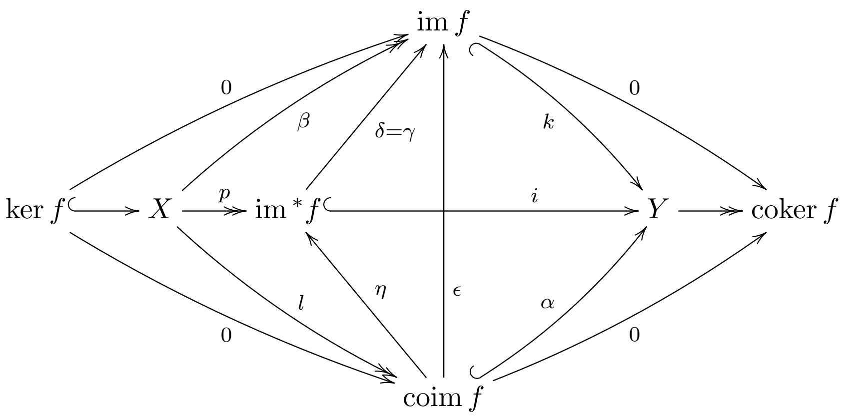
so, $q \circ \iota = 0$. By the universal property of the coimage, there is a unique $\gamma : \mathrm{coim}\, f \to K$ verifying $\gamma \circ p = q$. But then one gets $j \circ \gamma \circ p = j \circ q = f = i \circ p$ and since p is an epimorphism (by Lemma 1), it follows that $j \circ \gamma = i$ and then $\mathrm{coim}\, f$ satisfies the universal property of the weak image. Hence, $\mathcal{A}$ is hemispherical. □

Remark 3. *Given the usual isomorphism theorem, one has* $\mathrm{im}\, f \simeq \mathrm{coim}\, f$. *Consequently, one could have defined the weak image in abelian categories by the (usual) image, instead of the coimage.*

Corollary 2. *If $\mathcal{A}$ is an abelian category, then for every homomorphism f in $\mathcal{A}$, one has*

$$\mathrm{im}\, f \simeq \mathrm{im}^* f.$$

Proof. This follows directly from Theorem 1 and Théorème 5 from [1]. But we can make the isomorphism $\mathrm{im}\, f \xrightarrow{\sim} \mathrm{im}^* f$ more explicit. Given $f : X \to Y$, we have the following global diagram



where γ (resp. δ) is given by the universal property of $\mathrm{im}\, f$ (resp. $\mathrm{im}^* f$), ϵ is the isomorphism given by the usual Noether theorem and η is the isomorphism given by Theorem 1. As a matter of fact, one has $k \circ \delta = i = k \circ \epsilon \circ \eta^{-1}$ and since k is a monomorphism, we may conclude that $\delta = \epsilon \circ \eta^{-1}$. □

4. Normal and semi-normal monomorphisms in hemispherical categories

One might define the concept of "semi-normal monomorphism" or "semi-normal subobject" ("sous-truc" quoting Grothendieck [2]) in a hemispherical category ; a notion that slightly generalizes the concept of "normal monomorphisms" in pre-abelien categories. More precisely :

Definition 4. *A monomorphism $u : X \hookrightarrow Y$ in a hemispherical category $\mathcal{H}$ is said to be <u>semi-normal</u> if it admits a cokernel.*

Furthermore, given the choice of the "subobjects" of Y (see [2], section 1.1), a subobject X of Y is said to be <u>semi-normal</u> (and we denote $X \lhd Y$) if the natural embedding $X \overset{u}{\hookrightarrow} Y$ is a semi-normal monomorphism.

In this case, one may define the <u>generalized quotient</u> object $A//B$ by the formula

$$A//B := \mathrm{coker}\,(u).$$

We immediatly see that if $\mathcal{H} = \mathfrak{Grp}$ (resp. $\mathcal{H} = \mathfrak{Ann}_0$), this definition is indeed slightly more general than the one of normal subgroup (resp. with two-sided ideal).

Moreover, one has to pay attention not to confuse normal and semi-normal monomorphisms[2].

Remark 4. *Of course, in an abelian category, every monomorphism is semi-normal and normal and so, quotients and generalized quotients coincide.*

Example 3. *One has to be careful with this generalization. Indeed, a semi-normal monomorphism does not need to be normal. For instance, in $\mathfrak{Ann}_0$, the monomorphism*

$$\mathbb{Z} \hookrightarrow \mathbb{Q}$$

is semi-normal since the trivial pseudo-ring 0 *is a cokernel, but is obviously not normal.*

Nonetheless, Definition 4 is coherent in the following sense :

Proposition 2. *In every hemispherical category $\mathcal{H}$, a normal monomorphism is semi-normal.*

Proof. Let $u : X \hookrightarrow Y$ be a normal monomorphism. Then, there exists $\alpha : Y \to Z$ such that $u = \ker \alpha$. Let $p : Y \to \operatorname{coim} \alpha$ and $i : \operatorname{coim} \alpha \to Z$ be the natural morphisms. One has then $i \circ p \circ u = \alpha \circ u = 0 = i \circ 0$ and since i is supposed to be a monomorphism, this implies that $p \circ u = 0$. In another hand, let T be an object in $\mathcal{H}$ with a morphism $q : Y \to T$ such that $q \circ u = 0$. Then, by the universal property of the coimage, there is a unique $\gamma : \operatorname{coim} \alpha \to T$ satisfying $\gamma \circ p = q$. Hence, $\operatorname{coim} \alpha$ is a cokernel for u and so, u is semi-normal. □

5. Discussion about dualization and symmetry

The dual of the definitions of weak image and hemispherical category are defined straightforward. However, a natural question arises : since the definition of hemispherical category is not self dual, what goeas on if a category is hemispherical, as well as its opposite category ?

Definition 5. *Let $\mathcal{C}$ be a category and $f : X \to Y$ be a morphism in $\mathcal{C}$. We say that a pair (K, π) is a weak coimiage of f, and we denote $(K, \pi) = \operatorname{coim}^* f$ if*

1. *$\pi : X \to K$ is an epimorphism,*
2. *There exists $u : K \to Y$ such that $f = u \circ \pi$,*
3. *Forall pair (L, π') with π' an epi and $v : L \to Y$ such that $f = v \circ \pi'$, there exists a unique $\beta : L \to K$ such that $\pi' = \beta \circ \pi$*

From now, we say that a hemispherical category as define above is a left hemispherical category, for obvious reasons. The dual of this notion is the one of "right hemispherical category" :

Definition 6. *A category $\mathcal{H}$ is said to be right hemispherical if the following properties hold :*

1. *The exists a zero object in $\mathcal{H}$,*

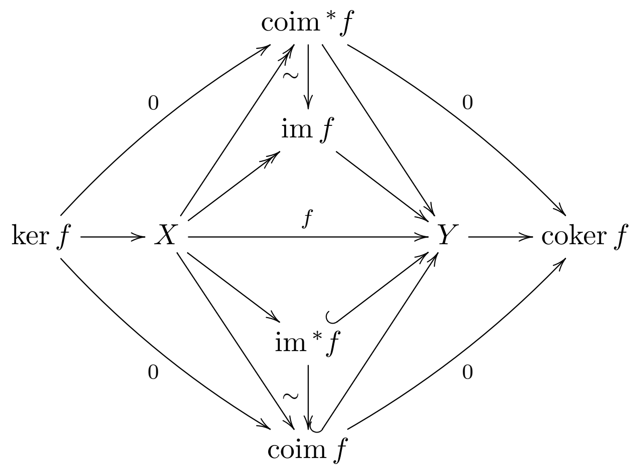
[2] A monomorphism is said to be normal if it is the kernel of some morphism.

(2) Every morphism in $\mathcal{H}$ admits a cokernel, an image and a weak coimage,

(3) For every morphism $f : X \to Y$, the natural morphism $X \to \operatorname{im} f$ is an epimorphism.

If a category $\mathcal{H}$ is left hemispherical and right hemispherical, then we say that $\mathcal{H}$ is spherical

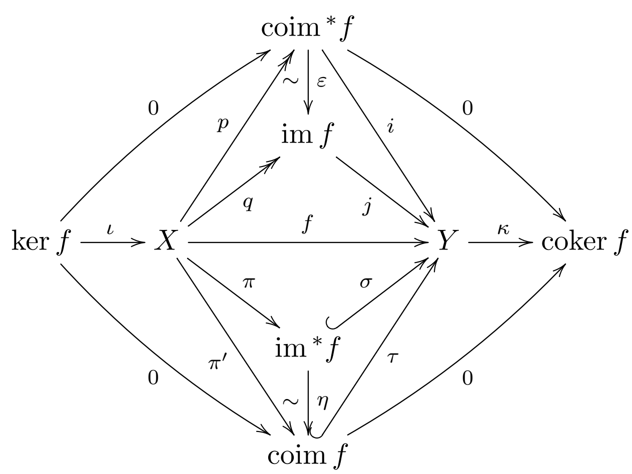
Let $\mathcal{H}$ be a spherical category. Then $\mathcal{H}$ admits a zero, (co)kernels, (co)images and weak (co)images. For a morphism $f : X \to Y$ in $\mathcal{H}$, one has then a global diagram



We have

$$\begin{cases} Y \twoheadrightarrow \operatorname{coker} f \\ \ker f \hookrightarrow X \end{cases} \Rightarrow \begin{cases} \operatorname{im} f \hookrightarrow Y \\ X \twoheadrightarrow \operatorname{coim} f \end{cases} \Rightarrow \begin{cases} \operatorname{coim}^* f \hookrightarrow Y \\ X \twoheadrightarrow \operatorname{im}^* f \end{cases}$$

because $\operatorname{coim}^* \simeq \operatorname{im}$ and $\operatorname{im}^* \simeq \operatorname{coim}$. We then have the following diagram



Since π' is epic, one has $\kappa \circ \tau = 0$ and by the universal property of the image, there exists a unique $\alpha : \operatorname{coim} f \to \operatorname{im} f$ such that $j \circ \alpha = \tau$. If $a, b : A \to \operatorname{coim} f$ such that $\alpha \circ a = \alpha \circ b$, then $\tau \circ a = j \circ \alpha \circ a = j \circ \alpha \circ b = \tau \circ b$ and, because τ is monic, this implies $a = b$, so α is monic.

Furthermore, since j is monic, and $j \circ \alpha \circ \pi = \tau \circ \pi = f = j \circ q$, we get $q = \alpha \circ \pi$. Dually, by the universal property of the coimage, there exists a unique $\beta : \operatorname{im} f \to \operatorname{coim} f$ such that $\beta \circ q = \pi$ and β is epic. Since q is epic and since $\tau \circ \beta \circ q = \tau \circ \pi = f = j \circ q$, then $\tau \circ \beta = j$. One has $\beta \circ \alpha \circ \pi = \beta \circ q = \pi$ and since π is epic, we have $\beta \circ \alpha = id_{\operatorname{coim} f}$ and dually, since j is monic and $j\alpha\beta = \tau\beta = j$, one gets $\alpha \circ \beta = id_{\operatorname{im} f}$. So, $\alpha : \operatorname{coim} f \to \operatorname{im} f$ is an isomorphism. Hence, we just have proved the following result :

Theorem 2. *Let $\mathcal{H}$ be an additive category. Then, $\mathcal{H}$ is spherical if and only if it is abelian.*

6. Discussion and Conclusion

The axiom system in the Definition 3 is not minimal. Indeed, recalling the proof of the Proposition 1, if $\mathcal{C}$ is a category satisfying all the axioms of Definition 3 except the one imposing weak images to exist in $\mathcal{C}$ and if $\operatorname{coim} f \to Y$ is a monomorphism for all f, then $\operatorname{coim} f$ satisfies the universal property of $\operatorname{im}^* f$; hence weak images do exist in $\mathcal{C}$. However, we chose here to keep the hypothesis of weak image's existence for hemispherical categories in order to get some "duality" in the definitions, namely the " weak duality" between coimage and weak image which are natural in some categories (like groups), although we must leave the kernel "dual-free". Furthermore, one may adapt some concepts concerning abelian categories to hemispherical categories. For instance, one could say that a short sequence of morphisms

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is <u>weakly exact</u> if $\operatorname{im}^* g = Z$, $\ker f = 0$ and $\ker g = \operatorname{im}^* f$. Going further in this analysis, we may say that a sequence of morphisms

$$\cdots \longrightarrow C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \longrightarrow \cdots$$

is a <u>weak complex</u> if $d^2 = 0$ and $\operatorname{im}^* d \lhd \ker d$. Then, we can define its <u>weak homology</u> :

$$^*H(C) := \ker d // \operatorname{im}^* d.$$

Then, natural questons arise

(1) Is this definition consitent ? Does it have good properties such as long (weak) exact sequence ?
(2) Is there a weak homotopy category ? Is it triangulated ? Does the concept of weak derived category make sense ?
(3) In case the category is abelian, does one have an isomorphism of functors $^*H = H$, or at least an isomorphism $^*H(C) \simeq H(C)$?
(4) Could one define the concept of semi-direct product, as in the category of groups ? Does this concept "measure" the default of the category to be abelian, just like groups ?
(5) Is there a concept of the Grothendieck group of an hemisphericl category ?

The first main interest of the weak image is to give a more intuitive definition of what an image could be in a category, trying to keep a weak notion of duality. The second one is to get an "isomorphism theorem" just as in abelian categories, avoiding the use of the cokernel, which does not exist in general, even in some usual categories (groups or pseudo-rings as we seen).

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