ANOTHER APPROACH TO CORRESPONDENCE FUNCTORS

SERGE BOUC AND JACQUES THÉVENAZ

Abstract. A new construction of correspondence functors is introduced, allowing for a more direct approach to the subject. In particular a description of the evaluations of simple functors is obtained by means of this approach, simplifying the proofs obtained earlier in [BT4].

1. Introduction

A correspondence functor is a functor from the category of finite sets and correspondences to the category of k-modules, where k is a fixed commutative ring. They are studied in [BT2, BT3, BT4]. A main tool is the correspondence functor F_T associated to any finite lattice T. In particular, F_T is used to prove some of the main properties of simple correspondence functors. Recall that, for any finite set X, the evaluation $F_T(X)$ is the free k-module whose basis is the set T^X of all maps from X to T. The simple correspondence functors are isomorphic to quotients of the (so-called) fundamental functors, which consequently play an important role in the theory.

It is proved in [BT3] that there is a perfect duality between F_T and $F_{T^{op}}$, where T^{op} denotes the opposite lattice. Moreover, the fundamental functor $\mathbb{S}_{E,R^{op}}$ appears as a quotient of F_T, while its dual $\mathbb{S}_{E,R}$ is isomorphic to a subfunctor of $F_{T^{op}}$. Here E is the set of join-irreducible elements of the lattice T, R is the natural partial order on E viewed as a full subposet of T, and R^{op} denotes the opposite partial order on E. It is also proved that the subfunctor $\mathbb{S}_{E,R} \subseteq F_{T^{op}}$ is generated by an explicit element $\gamma \in F_{T^{op}}(E)$. However, no further study of this subfunctor was carried out.

In the present paper, a new construction of correspondence functors is described directly inside the functor $F_{T^{op}}$, without using the duality. Swapping the role of T and T^{op}, we actually work inside a functor F_T. In particular, E is the set $\mathrm{Mirr}(T)$ of all meet-irreducible elements of T and R is the natural partial order on E. As an interesting aspect of these new constructions, some of the main results proved in [BT4] can be proved in a direct fashion, simplifying the whole approach to fundamental and simple correspondence functors.

We first construct a subfunctor Q_T of F_T, which can be characterized as the intersection of all the kernels of morphisms $F_T \to F_{T'}$ associated with surjective join-morphisms $T \to T'$ where $|T'| < |T|$. In order to describe the structure of Q_T, a k-basis of each evaluation $Q_T(X)$ at a finite set X is obtained by means of a k-linear idempotent endomorphism

$$F_T(X) \longrightarrow F_T(X)\,, \qquad \varphi \mapsto \widehat{\varphi}$$

whose image is equal to $Q_T(X)$.

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1.1. Theorem. *For any finite set X, the set*
$$\widehat{\mathcal{M}}_X = \{\, \widehat{\varphi} \mid \varphi \in T^X \text{ such that } E \subseteq \varphi(X) \,\}$$
is a k-basis of $Q_T(X)$.

The subfunctor Q_T contains in turn a subfunctor S_T, generated by the specific element $\widehat{i} \in F_T(E)$, where $i : E \to T$ denotes the inclusion map. This generator $\widehat{i}$ is precisely the element γ mentioned above. Again, the map $\varphi \mapsto \widehat{\varphi}$ is the key for a description of each evaluation $S_T(X)$.

1.2. Theorem. *For any finite set X, the set*
$$\widehat{\mathcal{B}}_X = \{\, \widehat{\varphi} \mid \varphi \in T^X \text{ such that } E \subseteq \varphi(X) \subseteq G \,\}$$
is a k-basis of $S_T(X)$.

Here G denotes a subset of T which plays a prominent role and which already was a main tool in [BT4]. The proof of this theorem is not easy, but it is much simpler than the proof of a similar result obtained in [BT4]. More precisely, we prove that S_T is isomorphic to the fundamental functor $\mathbb{S}_{E,R^{op}}$ and therefore we recover the description of a k-basis of each evaluation $\mathbb{S}_{E,R^{op}}(X)$, which was one of the main achievements in [BT4].

The parametrization of simple correspondence functors appears in [BT2, BT3, BT4]. In particular, we know from [BT4] that any simple correspondence functor is isomorphic to a quotient of a fundamental functor (but this requires some tedious technical work). Motivated by this fact, we introduce an additional construction $S_{T,V}$ and prove directly that this is a simple correspondence functor which is isomorphic to a quotient of S_T. Here V is a simple left $k\,\mathrm{Aut}(E,R)$-module and $S_{T,V} = S_T \otimes_{k\,\mathrm{Aut}(E,R)} V$ for a suitable right action of $\mathrm{Aut}(E,R)$ on S_T. This provides another approach to simple correspondence functors and their parametrization by triples (E,R,V).

The present development differs significantly from the previous procedure in [BT4] because it starts with the construction of S_T and rather quickly obtains a basis of each evaluation $S_T(X)$. This requires some subtle arguments, but involves less pain than in the previous approach. Only afterwards, S_T is shown to be isomorphic to a fundamental functor. Also, simple functors are obtained by a direct construction and it is rather straightforward to view them as quotients of fundamental functors. We see that the approach of the present paper has somehow inverted the main sequence of results. As an outcome, the whole exposition can be presented with a considerably shorter number of pages.

Whereas join-irreducible elements were prominent in our previous approach in [BT4], the present paper uses instead meet-irreducible elements in T. In both theorems above, E denotes the set $\mathrm{Mirr}(T)$ of all meet-irreducible elements of T, but we actually prove more generally that the same results hold if we use an arbitrary subset $E \subseteq \mathrm{Mirr}(T)$, replacing Q_T and S_T by more general subfunctors Q_T^E and S_T^E. The functor Q_T^E and its evaluations are studied in Section 3, while S_T^E appears in Section 4. The theorem on the evaluations of S_T^E is proved in Section 6, using an intermediate result in Section 5. The evaluation $S_T^E(E)$ of S_T^E at the set E is a minimal nonzero evaluation, which is studied in Section 7.

The next sections require to use $E = \mathrm{Mirr}(T)$ instead of a subset of $\mathrm{Mirr}(T)$. First the product of all restriction maps to $S_T(E)$ is proved to be injective in Section 8. This injectivity result is a main tool in Section 9, when we prove that S_T is isomorphic to the fundamental functor $\mathbb{S}_{E,R^{op}}$. The simple functors $S_{T,V}$ are constructed and classified in Section 10. Finally Section 11 explains why the parametrizaion of simple functors does not depend on the choice of a lattice T, but only on the poset (E,R).

Finally, in Sections 12, we explore the link between the approach of the present paper and the development obtained in [BT5], where some other correspondence functors are introduced by means of a suitable duality.

2. Preliminaries on finite lattices

Finite lattices are essential tools in our work on correspondence functors. In the present paper, meet-irreducible elements play a central role. We recall a few elementary facts and fix some notation. More details can be found in [BT3, BT4]. If T is a finite lattice, we write $\leq$ for its (partial) order relation, $\vee$ for its join, $\wedge$ for its meet, $\widehat{0}_T$, or simply $\widehat{0}$, for its bottom element and $\widehat{1}_T$, or simply $\widehat{1}$, for its top element. We also use the standard notation for intervals (closed intervals, half-open intervals, open intervals). Since T is finite, a *join-closed subset* U of T is a lattice with respect to the join of T and the induced meet $\widetilde{\wedge}$, defined by

$$u \,\widetilde{\wedge}\, u' = \bigvee_{\substack{t\in U\\ t\leq u\\ t\leq u'}} t\,.$$

Similarly, a *meet-closed subset* U of T is a lattice with respect to the meet of T and the induced join $\widetilde{\vee}$, defined by

$$u \,\widetilde{\vee}\, u' = \bigwedge_{\substack{t\in U\\ t\geq u\\ t\geq u'}} t\,.$$

An element of T is called *join-irreducible* if it cannot be written as the join of some subset of strictly smaller elements of T. In particular, $\widehat{0}$ is not join-irreducible because it is an empty join. We write $\mathrm{Jirr}(T)$ for the set of join-irreducible elements of T, viewed as a full subposet of T. For any $u \in T$, we define

$$r_T(u) = \bigvee_{\substack{t\in T\\ t<u}} t\,. \tag{2.1}$$

Then $r_T(u) = u$ if and only if $u \notin \mathrm{Jirr}(T)$, while, if $u \in \mathrm{Jirr}(T)$, $r_T(u)$ is the unique maximal element in $[\widehat{0}, u\,[$ and the open interval $]\,r_T(u), u\,[$ is empty.

The opposite lattice T^{op} of a finite lattice T is obtained by reversing the partial order, swapping $\vee$ and $\wedge$ and also swapping $\widehat{0}$ and $\widehat{1}$. An element of T is called *meet-irreducible* if it is join-irreducible in T^{op}, that is, if it cannot be written as the meet of some subset of strictly larger elements of T. In particular, $\widehat{1}$ is not meet-irreducible because it is an empty meet. We write $\mathrm{Mirr}(T)$ for the set of meet-irreducible elements of T, viewed as a full subposet of T. For any $u \in T$, we define

$$s_T(u) = \bigwedge_{\substack{t\in T\\ t>u}} t\,. \tag{2.2}$$

Then $s_T(u) = u$ if and only if $u \notin \mathrm{Mirr}(T)$, while, if $u \in \mathrm{Mirr}(T)$, $s_T(u)$ is the unique minimal element in $]\,u, \widehat{1}\,]$ and the open interval $]\,u, s_T(u)\,[$ is empty.

Recall that any $t \in T$ is a join of join-irreducible elements, hence

$$t = \bigvee_{\substack{e\in \mathrm{Jirr}(T)\\ e\leq t}} e\,,$$

and similarly

$$t = \bigwedge_{\substack{f \in \mathrm{Mirr}(T) \\ f \geq t}} f\,. \tag{2.3}$$

We shall work with the category having all finite lattices as objects and join-morphisms as morphisms. Recall that a map $\alpha : T \to T'$ between two finite lattices is a *join-morphism* if, for any (finite) subset U of T, we have

$$\alpha(\bigvee_{u \in U} u) = \bigvee_{u \in U} \alpha(u)$$

and in particular $\alpha(\widehat{0}) = \widehat{0}$ by using an empty join. Equivalently, α is a join-morphism if and only if $\alpha(t \vee u) = \alpha(t) \vee \alpha(u)$ for all $t, u \in T$ and $\alpha(\widehat{0}) = \widehat{0}$. A join-morphism is an order-preserving map, but we emphasize that, in general, it does not preserve meets. Consequently, the image $\mathrm{Im}(\alpha)$ of a join-morphism $\alpha : T \to T'$ is only join-closed in T' and may not be a sublattice of T'.

Let us recall the construction of opposite morphisms. If $\alpha : T \to T'$ is a join-morphism, there is a join-morphism $\alpha^{op} : T'^{op} \to T^{op}$, called the opposite of α, defined by

$$\alpha^{op}(t') = \bigvee_{\substack{t \in T \\ \alpha(t) \leq t'}} t\,.$$

In case α is surjective, we can also describe α^{op} by the condition

$$\alpha^{op}(t') = \bigvee_{\substack{t \in T \\ \alpha(t) = t'}} t = \sup\{t \in T \mid \alpha(t) = t'\}\,.$$

It is not hard to prove that α^{op} is a join-morphism from T'^{op} to T^{op} and that $(\alpha^{op})^{op} = \alpha$, see [BT3] for details.

Given $a \in \mathrm{Mirr}(T)$, it is elementary to check that $T - \{a\}$ is meet-closed, hence a lattice with respect to the induced join. Moreover, the map

$$f_a : T \to T - \{a\}\,, \qquad f_a(t) = \begin{cases} s_T(a) & \text{if } t = a\,, \\ t & \text{if } t \neq a\,, \end{cases} \tag{2.4}$$

is a join-morphism. This plays a important role in the next section.

3. Some subfunctors of the functor associated to a lattice

We first recall from [BT2, BT3, BT4] some basic definitions about correspondence functors. We denote by $\mathcal{C}$ the category of finite sets and correspondences. Its objects are the finite sets and the set $\mathcal{C}(Y, X)$ of morphisms from X to Y is the set of all correspondences from X to Y, namely all subsets of $Y \times X$ (using a reverse notation which is convenient for left actions). Correspondences can be composed in the usual manner. A correspondence from X to X is also called a relation on X and $R_X := \mathcal{C}(X, X)$ is the monoid of all relations on X. If k is a commutative ring, we let $k\mathcal{C}$ be the category whose objects are the finite sets and the set of morphisms from X to Y is the free k-module $k\mathcal{C}(Y, X)$ with basis $\mathcal{C}(Y, X)$. In particular, $k\mathcal{R}_X$ is the algebra of the monoid $\mathcal{R}_X$.

A *correspondence functor over* k is a functor from the category $\mathcal{C}$ to the category of k-modules, or in other words a k-linear functor from $k\mathcal{C}$ to the category of k-modules. If F is a correspondence functor and $U \subseteq Y \times X$ is a correspondence, we simply write

$$U\varphi := F(U)(\varphi)\,,\ \forall\, \varphi \in F(X)\,.$$

Associated to a finite lattice T, there is a correspondence functor F_T whose evaluation at a finite set X is the free k-module $F_T(X) = kT^X$ with basis the set T^X of all maps from X to T. The action of a correspondence $U \subseteq Y \times X$ on a map $\varphi \in T^X$ is the map in T^Y defined by

$$(U\varphi)(y) = \bigvee_{\substack{x \in X \\ (y,x) \in U}} \varphi(x)\,,\ \forall\, y \in Y\,. \tag{3.1}$$

Recall that if Z is a finite set and $S \subseteq Z \times Y$, then $S(U\varphi) = (SU)\varphi$ where SU denotes the usual composition of correspondences. This shows that F_T is indeed a functor.

Whenever $\pi : T \to T'$ is a join-morphism of finite lattices, there is an associated morphism of correspondence functors $\pi : F_T \to F_{T'}$ (still writen π for simplicity) which is defined, for every finite set X, by

$$\pi_X : F_T(X) \to F_{T'}(X)\,, \qquad \pi_X(\varphi) = \pi \circ \varphi\,,\ \forall\, \varphi \in T^X\,,$$

extended by k-linearity from this definition on basis elements.

From now on, we fix a finite lattice T with $|T| \geq 2$. Our main aim is to study some canonical subfunctors of F_T, the first being $\mathrm{Ker}(f_a)$ where $a \in \mathrm{Mirr}(T)$ and where $f_a : T \to T-\{a\}$ is the surjective join-morphism defined in (2.4). We emphasize that f_a also denotes the morphism of correspondence functors $f_a : F_T \to F_{T-\{a\}}$. Its kernel is of course a subfunctor of F_T.

3.2. Proposition. *Let $a \in \mathrm{Mirr}(T)$. For any finite set X, $\mathrm{Ker}(f_a)(X)$ has a k-basis*

$$\mathcal{A}_X = \{\, \varphi - f_a\,\varphi \mid \varphi \in T^X \ \textit{with}\ a \in \varphi(X)\,\}\,.$$

Proof : It is clear that $\varphi - f_a\,\varphi \in \mathrm{Ker}(f_a)(X)$ because f_a is idempotent. Notice that a does not belong to the image of $f_a\,\varphi$, so that φ and $f_a\,\varphi$ belong respectively to two disjoint subsets of the basis T^X of $F_T(X)$, namely $\{\varphi \in T^X \mid a \in \varphi(X)\}$ and $\{\psi \in T^X \mid a \notin \psi(X)\}$. Since the elements of the subset $\{\varphi \in T^X \mid a \in \varphi(X)\}$ are linearly independent, so are the elements of $\mathcal{A}_X$. It remains to prove that they generate $\mathrm{Ker}(f_a)(X)$.

Let $\sum_{\varphi \in T^X} \lambda_\varphi \varphi$ be an element of $\mathrm{Ker}(f_a)(X)$, where $\lambda_\varphi \in k$. Thus we have

$$\sum_{\varphi \in T^X} \lambda_\varphi (f_a\,\varphi) = 0\,. \tag{3.3}$$

For any $\psi \in T^X$ such that $a \notin \psi(X)$, define

$$Y_\psi = \{\varphi \in T^X \mid f_a\,\varphi = \psi\}\,.$$

Then (3.3) can be written

$$\sum_{\substack{\psi \in T^X \\ a \notin \psi(X)}} \Big(\sum_{\varphi \in Y_\psi} \lambda_\varphi \Big) \psi = 0\,.$$

By the linear independence of the functions in T^X, we deduce that

$$\sum_{\varphi \in Y_\psi} \lambda_\varphi = 0\,,\ \forall \psi \in T^X \text{ with } a \notin \psi(x)\,.$$

Notice that if $\varphi \in Y_\psi$ satisfies $a \notin \varphi(X)$, then $f_a\,\varphi = \varphi$, hence $\varphi = \psi$. Therefore $Y_\psi = \{\psi\} \sqcup Z_\psi$ where

$$Z_\psi = \{\varphi \in T^X \mid f_a\,\varphi = \psi \text{ and } a \in \varphi(X)\}\,.$$

It follows that we can write

$$0=\sum_{\varphi\in Y_\psi}\lambda_\varphi=\lambda_\psi+\sum_{\varphi\in Z_\psi}\lambda_\varphi\,,$$

and therefore $\lambda_\psi=-\sum_{\varphi\in Z_\psi}\lambda_\varphi$. Now we obtain

$$\sum_{\varphi\in T^X}\lambda_\varphi\varphi=\sum_{\substack{\psi\in T^X\\ a\notin\psi(X)}}\sum_{\varphi\in Y_\psi}\lambda_\varphi\varphi=\sum_{\substack{\psi\in T^X\\ a\notin\psi(X)}}\big(\lambda_\psi\psi+\sum_{\varphi\in Z_\psi}\lambda_\varphi\varphi\big)=\sum_{\substack{\psi\in T^X\\ a\notin\psi(X)}}\sum_{\varphi\in Z_\psi}\lambda_\varphi(\varphi-\psi)\,.$$

Since $\varphi\in Z_\psi$, we have $\psi=f_a\,\varphi$ and therefore the latter sum is a linear combination of elements of the form $\varphi-f_a\,\varphi$ with $a\in\varphi(X)$, as was to be shown. ∎

Our next main subfunctor of F_T is

$$Q_T=\bigcap_{a\in\mathrm{Mirr}(T)}\mathrm{Ker}(f_a)\,.$$

3.4. Remark. Let T' be a lattice with $|T'|<|T|$. It is not hard to prove that any surjective join-morphism $\pi:T\to T'$ is a composite ρf_a for some $a\in\mathrm{Mirr}(T)$, where $\rho:T-\{a\}\to T'$ is the restriction of π. Consequently $\mathrm{Ker}(f_a)\subseteq\mathrm{Ker}(\pi)$. Since any such $f_a:T\to T-\{a\}$ is a surjective join-morphism, it follows that

$$Q_T=\bigcap_{\pi}\mathrm{Ker}(\pi)\,,$$

where π runs over the set of all surjective join-morphisms from T to a lattice T' with $|T'|<|T|$. This is a pleasant description of Q_T, but we shall use only the previous definition.

Many results of this paper hold for a subset of $\mathrm{Mirr}(T)$ instead of the whole of $\mathrm{Mirr}(T)$. For simplicity of notation, we now fix a subset $E\subseteq\mathrm{Mirr}(T)$ and we use a superscript E whenever we need to emphasize a dependence on E. In particular, we define the subfunctor

(3.5)
$$Q_T^E=\bigcap_{a\in E}\mathrm{Ker}(f_a)\,.$$

Notice that $Q_T^{\varnothing}=F_T$, $Q_T^{\{a\}}=\mathrm{Ker}(f_a)$, and $Q_T^{\mathrm{Mirr}(T)}=Q_T$, as defined above.

For any subset A of $\mathrm{Mirr}(T)$, we also generalize the map s_T defined in (2.2) and introduce an operator $s_T^A:T\to T$ defined by

(3.6)
$$s_T^A(t)=\begin{cases}t & \text{if } t\notin A\,,\\ s_T(t) & \text{if } t\in A\,.\end{cases}$$

For every $\varphi\in T^X$, we now define a map $\varphi_A:X\to T$ as the composite

(3.7)
$$\varphi_A=s_T^A\circ\varphi\,,$$

or explicitly

$$\varphi_A(x)=\begin{cases}\varphi(x) & \text{if } \varphi(x)\notin A,\\ s_T(\varphi(x)) & \text{if } \varphi(x)\in A,\end{cases}$$

Notice that $\varphi_\varnothing=\varphi$ and $\varphi_{\{a\}}=f_{\{a\}}\varphi$ for any $a\in\mathrm{Mirr}(T)$. Next we define

(3.8)
$$\widehat{\varphi}^E=\sum_{A\subseteq E}(-1)^{|A|}\,\varphi_A\,.$$

Note that $\widehat{\varphi}^{\{a\}}=\varphi-f_{\{a\}}\varphi$ was used in Proposition 3.2. The element $\widehat{\varphi}^E$ belongs to $F_T(X)$ but we have more.

3.9. Lemma. *For any $\varphi \in T^X$, we have $\widehat{\varphi}^E \in Q_T^E(X)$.*

Proof : We need to prove that $f_a\,\widehat{\varphi}^E = 0$ for every $a \in E$. We first write

$$f_a\,\widehat{\varphi}^E = \sum_{\substack{A\subseteq E\\ a\in A}} (-1)^{|A|} f_a\,\varphi_A + \sum_{\substack{B\subseteq E\\ a\notin B}} (-1)^{|B|} f_a\,\varphi_B \;=\; \sum_{B\subseteq E-\{a\}} (-1)^{|B|} f_a(\varphi_B - \varphi_{B\cup\{a\}})\,,$$

and we claim that $f_a(\varphi_B - \varphi_{B\cup\{a\}}) = 0$ for any $B \subseteq E{-}\{a\}$.

To prove the claim, we now fix $x \in X$ and we consider the various possibilities for $\varphi(x)$. If $\varphi(x) \notin B \cup \{a\}$, then

$$f_a\,\varphi_B(x) = f_a\,\varphi(x) = f_a\varphi_{B\cup\{a\}}(x)\,.$$

If $\varphi(x) \in B$, then

$$f_a\,\varphi_B(x) = f_a\,s_T(\varphi(x)) = f_a\varphi_{B\cup\{a\}}(x)\,.$$

If $\varphi(x) = a$, then

$$f_a\,\varphi_B(x) = f_a\,\varphi(x) = f_a(a) = s_T(a) = f_a(s_T(a)) = f_a\,s_T(\varphi(x)) = f_a\,\varphi_{B\cup\{a\}}(x)\,.$$

This shows that $f_a\,\varphi_B = f_a\,\varphi_{B\cup\{a\}}$, proving the claim. □

In view of its important role in what follows, it is convenient to define

$$\mathcal{M}_X^E = \{\varphi \in T^X \mid E \subseteq \varphi(X)\}\,.$$

This comes into play in the following result.

3.10. Lemma. *Let $\varphi \in T^X$.*

(a) If $\varphi \notin \mathcal{M}_X^E$, then $\widehat{\varphi}^E = 0$.
(b) For any nonempty subset A of E, $\varphi_A \notin \mathcal{M}_X^E$. In particular $\widehat{\varphi_A}^E = 0$.

Proof : (a) Since $\varphi \notin \mathcal{M}_X^E$ by assumption, there exists $a \in E$ such that $a \notin \varphi(X)$. For any $B \subseteq E{-}\{a\}$ and any $x \in X$, we have

$$\varphi_{B\cup\{a\}}(x) = \begin{cases} \varphi(x) & \text{if } \varphi(x) \notin B \cup \{a\},\\ s_T(\varphi(x)) & \text{if } \varphi(x) = a,\\ s_T(\varphi(x)) & \text{if } \varphi(x) \in B.\end{cases}$$

The second case does not occur because $a \notin \varphi(X)$. Therefore

$$\varphi_{B\cup\{a\}}(x) = \begin{cases} \varphi(x) & \text{if } \varphi(x) \notin B,\\ s_T(\varphi(x)) & \text{if } \varphi(x) \in B.\end{cases}$$

But this is precisely the definition of $\varphi_B(x)$, so $\varphi_{B\cup\{a\}}(x) = \varphi_B(x)$. It follows that the two functions φ_B and $\varphi_{B\cup\{a\}}$ are equal and therefore cancel in the alternating sum defining $\widehat{\varphi}^E$. This holds for any $B \subseteq E{-}\{a\}$, so all the terms cancel in the alternating sum and $\widehat{\varphi}^E = 0$.

(b) Let a be a minimal element in A, so that a cannot have the form $a = s_T(b)$ for some $b \in A$. Then the definition of φ_A shows that a is not in the image of φ_A. Thus $\varphi_A \notin \mathcal{M}_X^E$, as required. Applying (a) to the function φ_A, we get in particular $\widehat{\varphi_A}^E = 0$. □

3.11. Corollary. *Let* $\widehat{\mathcal{M}}_X^E = \{\widehat{\varphi}^E \mid \varphi \in \mathcal{M}_X^E\}$. *The map*

$$\mathcal{M}_X^E \longrightarrow \widehat{\mathcal{M}}_X^E\,, \qquad \varphi \mapsto \widehat{\varphi}^E$$

is a bijection which preserves linear independence.

Proof : Since $\varphi_A \notin \mathcal{M}_X^E$ for every nonempty subset $A \subseteq E$, the definition of $\widehat{\varphi}^E$ shows that every $\widehat{\varphi}^E \in \mathcal{M}_X^E$ has the form

$$\widehat{\varphi}^E = \varphi + \varphi'\,,$$

where φ' is a linear combination of functions outside $\mathcal{M}_X^E$. Since the elements of $\mathcal{M}_X^E$ are linearly independent (because they form a subset of the basis T^X of $F_T(X)$), we deduce that so are the elements of $\widehat{\mathcal{M}}_X^E$. In particular, the map $\varphi \mapsto \widehat{\varphi}^E$ is a bijection. □

Our main result in this section provides an explicit description of the subfunctor Q_T^E.

3.12. Theorem. *For any finite set* X*, the set* $\widehat{\mathcal{M}}_X^E = \{\widehat{\varphi}^E \mid \varphi \in \mathcal{M}_X^E\}$ *is a* k*-basis of* $Q_T^E(X)$.

The proof is based on the following technical lemma.

3.13. Lemma. *Let* $\omega \in Q_T^E(X)$*,* $\omega \neq 0$*, and write* $\omega = \sum_{i\in I} \lambda_i\psi_i$ *for some index set* I*, with* $\psi_i \in T^X$*,* $\lambda_i \in k$*, and* $\lambda_i \neq 0$*. Then there exists* $i_0 \in I$ *such that* $\psi_{i_0} \in \mathcal{M}_X^E$.

Proof : Choose i_0 such that $|\psi_{i_0}(X) \cap E|$ is maximal among all $|\psi_i(X) \cap E|$, $i \in I$. Suppose that

$$a \notin \psi_{i_0}(X) \ \text{ for some } a \in E\,. \tag{3.14}$$

Since $\omega \in Q_T^E(X)$, we have $\omega \in \mathrm{Ker}(f_a)(X)$, so ω can be written as a linear combination of the basis

$$\mathcal{A}_X = \{\, \varphi - f_a\,\varphi \mid \varphi \in T^X \text{ with } a \in \varphi(X)\,\}\,,$$

see Proposition 3.2. Since ψ_{i_0} is one of the functions appearing in the given sum $\omega = \sum_{i\in I} \lambda_i\psi_i$, we see that ψ_{i_0} must be equal to some $f_a\,\varphi$ for some $\varphi \in T^X$ with $a \in \varphi(X)$ (noticing that it cannot be equal to φ because $a \in \varphi(X)$ while $a \notin \psi_{i_0}(X)$). But $f_a\,\varphi$ always comes together with some φ in the basis $\mathcal{A}_X$, so φ must also appear in the given expression of ω.

Now the functions φ and $f_a\,\varphi = \psi_{i_0}$ take the same values in $E-\{a\}$ whereas, in addition, φ also takes the value a. Thus $|\varphi(X) \cap E| = |\psi_{i_0}(X) \cap E| + 1$. This contradicts the maximality in the choice of i_0. Therefore our supposition (3.14) is impossible and so $E \subseteq \psi_{i_0}(X)$. Hence $\psi_{i_0} \in \mathcal{M}_X^E$. □

Proof of Theorem 3.12. By Corollary 3.11, we know that the elements of $\widehat{\mathcal{M}}_X^E$ are linearly independent. We need now to prove that they generate $Q_T^E(X)$. Let $\omega \in Q_T^E(X)$ and write

$$\omega = \sum_{\varphi\in T^X} \lambda_\varphi \varphi = \sum_{\varphi\in \mathcal{M}_X^E} \lambda_\varphi \varphi + \sum_{\psi\notin \mathcal{M}_X^E} \lambda_\psi \psi\,.$$

As in the proof of Corollary 3.11, we can write $\varphi = \widehat{\varphi}^E - \varphi'$ where φ' is a linear combination of functions outside $\mathcal{M}_X^E$. Thus we get

$$\omega = \sum_{\varphi\in \mathcal{M}_X^E} \lambda_\varphi(\widehat{\varphi}^E - \varphi') + \sum_{\psi\notin \mathcal{M}_X^E} \lambda_\psi \psi = \sum_{\varphi\in \mathcal{M}_X^E} \lambda_\varphi \widehat{\varphi}^E + \omega'\,,$$

where ω' is a linear combination of functions outside $\mathcal{M}_X^E$. Since $\omega \in Q_T^E(X)$ and $\widehat{\varphi} \in Q_T^E(X)$ by Lemma 3.9, we obtain $\omega' \in Q_T^E(X)$. By Lemma 3.13, ω' cannot be a linear combination of functions outside $\mathcal{M}_X^E$, unless $\omega' = 0$. It follows that $\omega = \sum_{\varphi\in\mathcal{M}_X^E} \lambda_\varphi \widehat{\varphi}^E$, proving that $\widehat{\mathcal{M}}_X^E$ generates $Q_T^E(X)$. ∎

3.15. Remark. A subfunctor H_T of F_T is introduced in [BT3] such that a k-basis of $H_T(X)$ is the set of all functions $\varphi : X \to T$ satisfying the condition $\mathrm{Jirr}(T) \nsubseteq \varphi(X)$. The subfunctor $Q_{T^{op}} = Q_{T^{op}}^{\mathrm{Mirr}(T^{op})}$ of the present paper is such that a k-basis of $Q_{T^{op}}(X)$ is the set of all functions $\varphi : X \to T^{op}$ satisfying the condition $\mathrm{Mirr}(T^{op}) = \mathrm{Jirr}(T) \subseteq \varphi(X)$. It is not hard to prove that H_T and $Q_{T^{op}}$ are orthogonal to each other with respect to the duality between F_T and $F_{T^{op}}$ mentioned in the introduction.

We continue with the fixed subset $E \subseteq \mathrm{Mirr}(T)$ and we write Σ_E for the symmetric group of all permutations of the set E.

3.16. Corollary.

(a) If $|X| < |E|$, then $Q_T^E(X) = \{0\}$.

(b) $Q_T^E(E)$ has a k-basis $\widehat{\mathcal{M}}_E^E = \{\,\widehat{i\sigma} \mid \sigma \in \Sigma_E\}$, where i denotes the inclusion map $E \to T$.

Proof : (a) Whenever $|X| < |E|$, the condition $E \subseteq \varphi(X)$ is impossible, so $\mathcal{M}_X^E$ is empty. Therefore $\widehat{\mathcal{M}}_X^E$ is empty too and $Q_T^E(X) = \{0\}$.

(b) When $X = E$, the condition $E \subseteq \varphi(X)$ implies that φ must map E onto E. This forces also the injectivity of φ and so $\varphi = i \circ \sigma$ for some $\sigma \in \Sigma_E$. Thus the basis $\widehat{\mathcal{M}}_E^E$ is as stated. ∎

We end this section with another property of Q_T^E. The definition of $\widehat{\varphi}^E$ in (3.8) extends k-linearly to a k-linear map $F_T(X) \to F_T(X)$ specified on the basis by $\varphi \mapsto \widehat{\varphi}^E$ and still written $\omega \mapsto \widehat{\omega}^E$ for any $\omega \in F_T(X)$.

3.17. Proposition. *The k-linear endomorphism $\omega \mapsto \widehat{\omega}^E$ of $F_T(X)$ is idempotent and its image is equal to $Q_T^E(X)$. In particular, $\widehat{\omega}^E = \omega$ for every $\omega \in Q_T^E(X)$.*

Proof : Let $\varphi \in T^X$. By Lemma 3.10, $\widehat{\varphi_A}^E = 0$ for any nonempty subset A of E. It follows that

$$\widehat{\widehat{\varphi}^E}^E = \sum_{A\subseteq E} (-1)^{|A|}\, \widehat{\varphi_A}^E = \widehat{\varphi_\varnothing}^E = \widehat{\varphi}^E\,.$$

Extending this by k-linearity, we see that the map $\omega \mapsto \widehat{\omega}^E$ is idempotent. Since $\widehat{\varphi}^E$ belongs to $Q_T^E(X)$ if $\varphi \in \mathcal{M}_X^E$ (by Lemma 3.9) and is zero otherwise (by Lemma 3.10), the image of the idempotent is exactly $Q_T^E(X)$ (by Theorem 3.12). ∎

It should be noted that the image of the idempotent happens to be a subfunctor of F_T, but the idempotent map is not a morphism of functors.

4. More subfunctors

We now introduce more subfunctors of F_T which will be our main objects of study. We let T be a finite lattice with $|T| \geq 2$ and we let E be a fixed subset of $\mathrm{Mirr}(T)$, as before. Since E is fixed, we can simplify notation and write simply $\widehat{\varphi}$ instead of $\widehat{\varphi}^E$, as defined by (3.8). We let S_T^E be the subfunctor of F_T generated by $\widehat{i} \in F_T(E)$, where i denotes the inclusion map $E \to T$. Explicitly, for any finite set X, $S_T^E(X)$ is k-linearly generated by all elements of $F_T(X)$ of the form $U\,\widehat{i}$ where $U \subseteq X \times E$ is a correspondence. Since $\widehat{i} \in Q_T^E(E)$ by Lemma 3.9, S_T^E is a subfunctor of Q_T^E.

We need to study $U\widehat{i}$. Since this belongs to $Q_T^E(X)$, it can be expressed in the basis $\widehat{\mathcal{M}}_X^E = \{\widehat{\varphi} \mid \varphi \in T^X,\ E \subseteq \varphi(X)\}$ (see Theorem 3.12).

4.1. Lemma. *Let $U \subseteq X \times E$ be a correspondence. Then*

$$U\,\widehat{i} = \sum_{A \subseteq E} (-1)^{|A|}\, Ui_A = \sum_{A \subseteq E} (-1)^{|A|}\, \widehat{Ui_A} = \sum_{\substack{A \subseteq E \\ Ui_A \in \mathcal{M}_X}} (-1)^{|A|}\, \widehat{Ui_A}\,.$$

Proof : The first equality follows from the definition of $\widehat{i}$ and k-linearity. Since $U\,\widehat{i}$ belongs to $Q_T^E(X)$ (because Q_T^E is a subfunctor), it is fixed under the map $\omega \mapsto \widehat{\omega}$ (by Proposition 3.17), proving the second equality. Finally $\widehat{Ui_A} = 0$ if $Ui_A \notin \mathcal{M}_X^E$ (by Lemma 3.10), proving the third equality. □

In this section and the next, our purpose will be to prove that $S_T^E(X)$ has a basis

$$\textbf{(4.2)} \qquad \widehat{\mathcal{B}}_X^E = \{\widehat{\varphi} \mid \varphi \in \mathcal{B}_X^E\}\,,$$

where $\widehat{\varphi} = \widehat{\varphi}^E$ and $\mathcal{B}_X^E = \{\varphi \in T^X \mid E \subseteq \varphi(X) \subseteq G^E\}$, for a suitable subset G^E of T, which we need to define. Our previous approach to simple correspondence functors in [BT4] used a suitable lattice T', the set $E = \mathrm{Jirr}(T')$, the subset $\bigwedge E$ of all meets of elements of E and a certain subset $G = \bigwedge E \sqcup \widetilde{G}$. Since we have swapped the role of T and T^{op}, we need here the opposite version of this construction. We start with an arbitrary subset $E \subseteq \mathrm{Mirr}(T)$ and we let $\bigvee E$ be the set of all joins of elements of E. For any $t \in T$, we define

$$\textbf{(4.3)} \qquad \rho(t) = \bigvee_{\substack{e \in E \\ e < t}} e \quad \text{and inductively} \quad \rho^j(t) = \rho\rho^{j-1}(t) \ \text{ for any } j > 1\,,$$

and we set $\rho^\infty(t) = \rho^n(t)$ where n is such that $\rho^n(t) = \rho^{n+1}(t)$. Notice that ρ depends on E and should be written ρ^E, but we keep the notation simple for the clarity of reading. Using the map s_T^E defined in (3.6), we write inductively $(s_T^E)^j(t) = s_T^E(s_T^E)^{j-1}(t)$ for any $j > 1$, and $(s_T^E)^\infty(t) = (s_T^E)^n(t)$ where n is such that $(s_T^E)^n(t) = (s_T^E)^{n+1}(t)$. Then we define

$$\textbf{(4.4)} \qquad G^E = \bigvee E \sqcup \widetilde{G}^E \ \text{ where } \ \widetilde{G}^E = \{h \in T \mid (s_T^E)^\infty \rho^\infty(h) = h,\ h \notin \bigvee E\}\,.$$

As the set $\widetilde{G}^E$ plays a prominent role throughout this paper, it is important to note its properties. The following lemma is Lemma 2.9 in [BT4], applied to the opposite lattice T^{op} and to the fixed subset $E \subseteq \mathrm{Mirr}(T)$.

4.5. Lemma. *For any $h \in \widetilde{G}^E$, there is a unique chain*

$$b_n < b_{n-1} < \ldots < b_1 < b_0 = h \qquad (\textit{with } n \geq 1)$$

such that

(a) $s_T^E(b_{i+1}) = b_i$ for $0 \leq i \leq n-1$ and $s_T^E(b_0) = b_0$.
(b) $\rho(b_i) = b_{i+1}$ for $0 \leq i \leq n-1$ and $\rho(b_n) = b_n$.

Part (a) implies that $b_1, \ldots, b_n$ belong to E, but $b_0 = h \notin E$. In particular $b_0 = (s_T^E)^\infty(b_n)$. Also, (b) implies that $\rho^\infty(b_0) = b_n$. Note that the condition $h \notin \bigvee E$ forces $\rho(h) < h$, that is $b_1 < b_0$, so that $n \geq 1$.

We write $k\widehat{\mathcal{B}}_X^E$ for the k-linear span of the set $\widehat{\mathcal{B}}_X^E$ defined in (4.2). Notice that $\widehat{\mathcal{B}}_X^E$ is a k-linearly independent set, because it is a subset of the basis $\widehat{\mathcal{M}}_X^E$ of $Q_T^E(X)$.

4.6. Theorem. *For any finite set X, $S_T^E(X) \subseteq k\widehat{\mathcal{B}}_X^E$.*

We first need a technical lemma.

4.7. Lemma. *Let $x \in X$ and $U \subseteq X \times E$. Suppose that $\{e \in E \mid (x,e) \in U\}$, viewed as a subposet of E, has at least two maximal elements. Then $(Ui)(x) = (Ui_A)(x)$ for every subset $A \subseteq E$. Moreover $(Ui)(x) > e$ for every element e of the poset $\{e \in E \mid (x,e) \in U\}$.*

Proof : Let f, g be two distinct maximal elements of $\{e \in E \mid (x,e) \in U\}$. Then, by Definition 3.1,

$$(Ui)(x) = \bigvee_{(x,e)\in U} i(e) = \bigvee_{(x,e)\in U} e \geq f, g\,.$$

Therefore $(Ui)(x) \geq f \vee g > f$ and so $(Ui)(x) \geq s_T^E(f) > f$. Similarly $(Ui)(x) \geq s_T^E(g) > g$ for every maximal element g of the poset $\{e \in E \mid (x,e) \in U\}$.

If $e \in \{e \in E \mid (x,e) \in U\}$ and $g \in E$ is maximal with $e \leq g$, then $s_T^E(e) \leq s_T^E(g) \leq (Ui)(x)$. Therefore

$$(Ui_E)(x) = \bigvee_{(x,e)\in U} i_E(e) = \bigvee_{(x,e)\in U} s_T^E(e) \leq (Ui)(x)\,.$$

It follows that, for every subset $A \subseteq E$, we get

$$(Ui)(x) \leq (Ui_A)(x) \leq (Ui_E)(x) \leq (Ui)(x)\,,$$

hence $(Ui_A)(x) = (Ui)(x)$. ∎

Proof of Theorem 4.6. Since $S_T^E(X)$ is k-linearly generated by all elements of $F_T(X)$ of the form $U\,\widehat{i}$ where $U \subseteq X \times E$, we need to prove that $U\,\widehat{i}$ belongs to $k\widehat{\mathcal{B}}_X^E$. In the expression

$$U\,\widehat{i} = \sum_{\substack{A\subseteq E\\ Ui_A\in\mathcal{M}_X}} (-1)^{|A|}\, \widehat{Ui_A}$$

of Lemma 4.1, we will show that the function Ui_A belongs to $\mathcal{B}_X^E$, that is, $E \subseteq (Ui_A)(X) \subseteq G^E$, where G^E is defined by (4.4). Since $Ui_A \in \mathcal{M}_X^E$, we know that $E \subseteq (Ui_A)(X)$. We let $x \in X$ and we need to show that $(Ui_A)(x) \in G^E$.

If $\{e \in E \mid (x,e) \in U\}$ has at least two maximal elements, then by Lemma 4.7

$$(Ui_A)(x) = (Ui)(x) = \bigvee_{(x,e)\in U} e \in \bigvee E \subseteq G^E\,.$$

If $\{e \in E \mid (x,e) \in U\}$ is empty, then $(Ui_A)(x)$ is an empty join (i.e. $\widehat{0}$), which belongs to $\bigvee E$, hence to G^E. So we can assume that $\{e \in E \mid (x,e) \in U\}$ has a unique maximal element a. Then

$$(Ui_A)(x) = \bigvee_{(x,e)\in U} i_A(e) = \begin{cases} a & \text{if } a \notin A\,,\\ s_T(a) & \text{if } a \in A\,.\end{cases}$$

Now $a \in E \subseteq \bigvee E \subseteq G^E$, so we are left to prove that $s_T(a) \in G^E$. This is again clear if $s_T(a) \in \bigvee E$, so we now assume that $s_T(a) \notin \bigvee E$ for some $a \in A$ and we have to prove that $s_T(a) \in \widetilde{G}^E$.

Let $b = b_0 = s_T(a)$ and $b_1 = a$. We have $\rho(b) \leq b$, $\rho(b) \in \bigvee E$, and $b \notin \bigvee E$, hence $\rho(b) < b$. Now $b = s_T(b_1)$, so the interval $]b_1, b[$ is empty. Thus the strict inequality $b_1 \leq \rho(b) < b$ forces $\rho(b) = b_1$. For any integer $i \geq 1$, we set $b_i = \rho^i(b)$ and we let n be minimal such that $\rho^n(b) = \rho^\infty(b)$, so that we have a chain

$$b_n < b_{n-1} < \ldots < b_1 < b_0 = b \;\text{ with } \rho(b_i) = b_{i+1} \; (0 \leq i \leq n-1)\,.$$

We now prove by induction that

$$\textbf{(4.8)} \qquad b_{r+1} \in A \;\text{ and }\; s_T(b_{r+1}) = b_r \;\text{ for }\; 0 \leq r \leq n-1\,.$$

This is clear if $r = 0$ because $b_1 = a \in A$ and $s_T(b_1) = s_T(a) = b_0$. So assume that $b_i \in A$ and $s_T(b_i) = b_{i-1}$ for some $1 \leq i \leq n-1$. Since $Ui_A \in \mathcal{M}_X^E$, we have

$$b_i \in A \subseteq E \subseteq (Ui_A)(X)\,,$$

hence $b_i = (Ui_A)(x_i)$ for some $x_i \in X$. Let $E_i = \{e \in E \mid (x_i, e) \in U\}$. If E_i had at least two maximal elements, then Lemma 4.7 would imply

$$b_i = (Ui_A)(x_i) = (Ui)(x_i) = \bigvee_{e \in E_i} e\,,$$

with each such e strictly smaller than b_i. Therefore we would get $\rho(b_i) = b_i$ contrary to the fact that $\rho(b_i) = b_{i+1}$ (since $i \leq n-1$). If now E_i was empty, then we would obtain $b_i = \widehat{0}$, contrary to the fact that $\rho(b_i) = b_{i+1} < b_i$. It follows that E_i has a unique maximal element c_i and we get

$$b_i = (Ui_A)(x_i) = \bigvee_{e \in E_i} i_A(e) = i_A(c_i) = \begin{cases} c_i & \text{if } \; c_i \notin A\,, \\ s_T(c_i) & \text{if } \; c_i \in A\,. \end{cases}$$

The case $c_i \notin A$ is impossible because we would get $b_i = c_i$, but we know that $b_i \in A$ (by induction). Thus $c_i \in A$ and $b_i = s_T(c_i)$. Now we have $c_i \in E$ and $c_i < b_i$, so that

$$c_i \leq \bigvee_{\substack{f \in E \\ f < b_i}} f = \rho(b_i) = b_{i+1} < b_i = s_T(c_i)\,.$$

Since the interval $]c_i, s_T(c_i)[$ is empty, we obtain $c_i = \rho(b_i) = b_{i+1}$. Therefore $s_T(b_{i+1}) = s_T(c_i) = b_i$ as required for the proof of (4.8). Moreover, we have seen that $b_{i+1} = c_i \in A$, completing the induction proof of (4.8). Note also that $s_T(b_0) = b_0$ because $b_0 = b \notin E$.

Now Definition 3.6 implies that $s_T(b_i) = s_T^E(b_i)$ when $0 \leq i \leq n-1$, because $b_i \in E$. So we get a chain

$$b_n < b_{n-1} < \ldots < b_1 < b_0 = b \;\text{ with } \rho(b_i) = b_{i+1} \text{ and } s_T^E(b_{i+1}) = b_i \; (0 \leq i \leq n-1)\,.$$

Therefore $b = (s_T^E)^\infty \rho^\infty(b)$. Since we started with $b = s_T(a) \notin \bigvee E$, we obtain $s_T(a) \in \widetilde{G}^E$, as required. This completes the proof of Theorem 4.6. $\square$

5. The element $\widehat{\theta}$

By Theorem 4.6, we know that $S_T^E(X) \subseteq k\widehat{\mathcal{B}}_X^E$. Our main task in Section 6 will be to show that the opposite inclusion also holds. In other words, we will need to prove that $\widehat{\varphi} \in S_T^E(X)$ for any function $\varphi \in T^X$ such that $E \subseteq \varphi(X) \subseteq G^E$, where we write again $\widehat{\varphi} = \widehat{\varphi}^E$ for simplicity of notation. In this section, we prove that this holds for the specific element $\widehat{\theta}$, where $\theta : G^E \to T$ denotes the inclusion map and $G^E = \bigvee E \sqcup \widetilde{G}^E$ as in (4.4). In fact, $\widehat{\theta}$ plays a universal role.

The proof that $\widehat{\theta} \in S_T^E(G^E)$ uses a subset P of E which we now define. By Lemma 4.5, for each $h \in \widetilde{G}^E$, there is a totally ordered subset

$$b_n < b_{n-1} < \ldots < b_1 < b_0 = h \text{ with } \rho(b_i) = b_{i+1} \text{ and } s_T^E(b_{i+1}) = b_i \ (0 \le i \le n-1)$$

and we write $P_h = \{b_n, b_{n-1}, \ldots, b_1\}$. Since $s_T^E(b_i) = b_{i-1} > b_i$, we have $b_i \in E$ for each $1 \le i \le n$, hence $P_h \subseteq E$. However, the top element $b_0 = h$ does not belong to E. Note that $s_T^E(\rho(b_i)) = b_i$ for $0 \le i \le n-1$, while the bottom element $b_n = \rho^\infty(h)$ is in P_h and is fixed under ρ. We then define

$$P = \bigsqcup_{h \in \widetilde{G}^E} P_h \tag{5.1}$$

and we note that this is a disjoint union because any $b_i \in P_h$ determines uniquely h via $h = (s_T^E)^i(b_i) = (s_T^E)^\infty(b_i)$. We also set

$$P^\infty = \rho^\infty(P) = \{\rho^\infty(h) \mid h \in \widetilde{G}^E\} \tag{5.2}$$

and we recall that $\rho^\infty(h)$ is the bottom element b_n in the notation above. Finally, for any subset $C \subseteq P$, we define a correspondence

$$U_C = \{(g,e) \in G^E \times E \mid e < g\} \sqcup \{(e,e) \mid e \in E{-}C\} \subseteq G^E \times E\,,$$

which we can also write as

$$U_C = \{(g,e) \mid g \in (G^E{-}E) \sqcup C\,,\ e{<}g\} \cup \{(g,e) \mid g \in E{-}C\,,\ e{\le}g\}\,. \tag{5.3}$$

Note that $C \subseteq D$ implies $U_C \supseteq U_D$.

5.4. Theorem. *Let $\theta : G^E \to T$ be the inclusion map. Then*

$$\widehat{\theta} = \sum_{C \subseteq P} (-1)^{|C|}\, U_C\, \widehat{i}\,.$$

In particular, $\widehat{\theta}$ belongs to $S_T(G^E)$.

The proof is rather involved. We will start with the right hand side and use several reductions to show that it is equal to $\widehat{\theta}$. We first decompose $U_C\,\widehat{i}$ according to Lemma 4.1:

$$U_C\,\widehat{i} = \sum_{A \subseteq E} (-1)^{|A|}\, \widehat{U_C i_A}\,. \tag{5.5}$$

We need to describe the values of the function $U_C i_A : G^E \to T$. The set $G^E = \bigvee E \sqcup \widetilde{G}^E$ is a disjoint union

$$G^E = (E{-}C) \sqcup C \sqcup (\bigvee E{-}E) \sqcup \widetilde{G}^E\,,$$

and we will consider the cases successively.

5.6. Lemma. *Let $C \subseteq P$ and $A \subseteq E$. Let $g \in G^E$.*

(a) *If $g \in E{-}C$, then* $(U_C i_A)(g) = \begin{cases} g & \text{if } g \notin A\,, \\ s_T(g) & \text{if } g \in A\,. \end{cases}$

(b) *If $g \in C$ and $g \notin P^\infty$, then* $(U_C i_A)(g) = \begin{cases} \rho(g) & \text{if } \rho(g) \notin A\,, \\ g & \text{if } \rho(g) \in A\,. \end{cases}$

(c) *If $g \in C$ and $g \in P^\infty$, then $(U_C i_A)(g) = g$.*

(d) *If $g \in \bigvee E{-}E$, then $(U_C i_A)(g) = g$.*

(e) *If $g \in \widetilde{G}^E$, then* $(U_C i_A)(g) = \begin{cases} \rho(g) & \text{if } \rho(g) \notin A\,, \\ g & \text{if } \rho(g) \in A\,. \end{cases}$

Moreover, in all cases (b)-(e), we have $(U_C i_A)(g) = \bigvee_{e<g} i_A(e)$.

Proof : Recall that $(U_C i_A)(g) = \bigvee_{\substack{e\in E\\ (g,e)\in U_C}} i_A(e)$.

(a) Let $g \in E - C$. The definition of U_C shows that $(g,g) \in U_C$, so that g is the unique maximal element of $\{e \in E \mid (g,e) \in U_C\}$. Therefore

$$(U_C i_A)(g) = i_A(g) = \begin{cases} g & \text{if } \ g \notin A\,, \\ s_T(g) & \text{if } \ g \in A\,. \end{cases}$$

(b) Let $g \in C$ with $g \notin P^\infty$. The definition of U_C shows that $(g,e) \in U_C$ if and only if $e < g$, hence $(U_C i_A)(g) = \bigvee_{e<g} i_A(e)$. Since P is a disjoint union of totally ordered subsets and $g \notin P^\infty$, $\rho(g)$ is the unique maximal element of $\{e \in E \mid e < g\}$ and $s_T(\rho(g)) = g$. Therefore

$$(U_C i_A)(g) = i_A(\rho(g)) = \begin{cases} \rho(g) & \text{if } \ \rho(g) \notin A\,, \\ s_T(\rho(g)) = g & \text{if } \ \rho(g) \in A\,. \end{cases}$$

(c) Let $g \in C$ with $g \in P^\infty$. Then $\rho(g) = g$. Moreover, $e < g$ implies $g \geq s_T(e) \geq i_A(e) \geq e$ and therefore

$$g \geq \bigvee_{e<g} s_T(e) \geq \bigvee_{e<g} i_A(e) \geq \bigvee_{e<g} e = \rho(g) = g\,.$$

It follows that $(U_C i_A)(g) = \bigvee_{e<g} i_A(e) = g$.

(d) Let $g \in \bigvee E - E$. The definition of U_C implies that $(U_C i_A)(g) = \bigvee_{e<g} i_A(e)$. Moreover, g is a join of elements of E, all strictly smaller than g since $g \notin E$. Thus $g = \bigvee_{e<g} e$. We obtain again

$$g \geq \bigvee_{e<g} s_T(e) \geq \bigvee_{e<g} i_A(e) \geq \bigvee_{e<g} e = g\,,$$

so that $(U_C i_A)(g) = \bigvee_{e<g} i_A(e) = g$.

(e) Let $g \in \widetilde{G}^E$. The definition of U_C implies that $(U_C i_A)(g) = \bigvee_{e<g} i_A(e)$. Moreover, $\rho(g)$ is in P, hence in particular in E, so that $\rho(g)$ is the unique maximal element of $\{e \in E \mid e < g\}$. It follows that

$$(U_C i_A)(g) = i_A(\rho(g)) = \begin{cases} \rho(g) & \text{if } \ \rho(g) \notin A\,, \\ s_T(\rho(g)) = g & \text{if } \ \rho(g) \in A\,. \end{cases}$$

This completes the proof. □

We now show that many terms $\widehat{U_C i_A}$ vanish.

5.7. Lemma. *Let* $C \subseteq P$ *and* $A \subseteq E$. *If* $A \nsubseteq P$, *then* $\widehat{U_C i_A} = 0$.

Proof : By Lemma 3.10, all we have to show is that $U_C i_A \notin \mathcal{M}^E_{G^E}$, that is, $E \nsubseteq (U_C i_A)(G^E)$. Since $A \nsubseteq P$, we can choose $a \in A$ minimal such that $a \notin P$ and we claim that $a \notin (U_C i_A)(G^E)$. Looking for a contradiction, we assume that $a = (U_C i_A)(g)$ for some $g \in G^E$ and we treat the cases of Lemma 5.6.

(a) Suppose $g \in E - C$. If $g \notin A$, Lemma 5.6 asserts that $a = g$, hence $a \notin A$, a contradiction. If $g \in A$, we get $a = s_T(g) > g$ and by minimality of a, we deduce that $g \in P$. Since P is a disjoint union of totally ordered subsets, we must have $s_T(g) \in P$ or $s_T(g) \in \widetilde{G}^E$ by Lemma 4.5. The case $a = s_T(g) \in P$ is impossible because $a \notin P$. The case $a = s_T(g) \in \widetilde{G}^E$ is impossible because the elements of $\widetilde{G}^E$ are not in E, while $a \in A \subseteq E$.

(b) Suppose $g \in C$ and $g \notin P^\infty$. The case $a = \rho(g)$ cannot hold because $a \in A$ and $\rho(g) \notin A$. The case $a = g$ cannot hold either because $g \in C$, hence $g \in P$, whereas $a \notin P$.

(c) Suppose $g \in C$ and $g \in P^\infty$. Then $a = g$ and this contradicts again the fact that $g \in P$ and $a \notin P$.

(d) Suppose $g \in \bigvee E - E$. We obtain again $a = g$. This is a contradiction because $g \notin E$ and $a \in A \subseteq E$.

(e) Suppose $g \in \widetilde{G}^E$. The first case $a = \rho(g)$ cannot hold because $a \in A$ and $\rho(g) \notin A$. The second case $a = g$ cannot hold either because $g \in \widetilde{G}^E$ and this is not in E while $a \in E$.

We have obtained a contradiction in all cases, proving the claim $a \notin (U_C i_A)(G^E)$. Therefore $E \not\subseteq (U_C i_A)(G^E)$ and this completes the proof of Lemma 5.7. □

5.8. Corollary. $U_C\,\widehat{i} = \displaystyle\sum_{A \subseteq P} (-1)^{|A|}\,\widehat{U_C i_A}$.

Proof : Instead of running over all subsets A of E, the sum (5.5) can be restricted to run only over all subsets A of P, by Lemma 5.7. □

We will now consider the alternating sum over C as in the statement of Theorem 5.4 and we want to show that further cancelations occur. To this end, we fix a subset A of P which we write $A = \bigcup_{h \in \widetilde{G}^E} A_h$ where $A_h = A \cap P_h$. Whenever $P_h - A_h$ is nonempty, we let d_h be the smallest element of $P_h - A_h$ (which exists because P_h is totally ordered). Then we define

$$D = \{d_h \mid h \in \widetilde{G}^E \text{ such that } P_h - A_h \neq \varnothing\}\,.$$

5.9. Lemma. *Let $A \subseteq P$ and let D be defined as above. If C is a subset of P and $S = C - D$, then $U_C i_A = U_S i_A$.*

Proof : Again we need to treat the cases of Lemma 5.6. Let $g \in G^E$. Assume first that $g \in E - C$. Since $S \subseteq C$, g also belongs to $E - S$. Case (a) of Lemma 5.6 applies and we see that $(U_C i_A)(g) = (U_S i_A)(g)$.

Suppose now that $g \in S$, so that $g \in C$. The additional statement in Lemma 5.6 yields $(U_C i_A)(g) = \bigvee_{e<g} i_A(e) = (U_S i_A)(g)$.

Next assume that $g \in E - S$ and $g \in C$. Since $S = C - D$, we deduce that $g \in D$ and in particular $g \notin A$. Then Case (a) of Lemma 5.6 implies that $(U_S i_A)(g) = g$ because $g \notin A$. Since $g \in D$, we can write $g = d_h$ and d_h is the smallest element of $P_h - A_h$, for some $h \in \widetilde{G}^E$ (namely $h = (s_T^E)^\infty(g)$). For the computation of $(U_C i_A)(g)$, we note that $g \in C$, so either (b) or (c) in Lemma 5.6 applies, depending on whether or not $g \in P^\infty$.

If $g \notin P^\infty$, we have $\rho(g) < g$ and both lie in the same totally ordered set P_h. Since $g = d_h$ is the smallest element of $P_h - A_h$, we must have $\rho(g) \in A_h$ and in particular $\rho(g) \in A$. By Case (b) in Lemma 5.6, we see that $(U_C i_A)(g) = g$. If now $g \in P^\infty$, then Case (c) in Lemma 5.6 implies that $(U_C i_A)(g) = g$. This shows that $(U_C i_A)(g) = g = (U_S i_A)(g)$.

Finally, if $g \in \bigvee E - E$ or if $g \in \widetilde{G}^E$, then $(U_C i_A)(g)$ and $(U_S i_A)(g)$ are equal because they are both equal to $\bigvee_{e<g} i_A(e)$. □

5.10. Corollary. $\displaystyle\sum_{C\subseteq P}(-1)^{|C|}\,U_C\,\widehat{i}=(-1)^{|P|}\sum_{C\subseteq P}(-1)^{|C|}\,\widehat{U_Ci_P}\,.$

Proof : By Corollary 5.8,

$$\sum_{C\subseteq P}(-1)^{|C|}\,U_C\,\widehat{i}=\sum_{\substack{C\subseteq P\\ A\subseteq P}}(-1)^{|C|+|A|}\,\widehat{U_Ci_A}=\sum_{A\subseteq P}(-1)^{|A|}\sum_{C\subseteq P}(-1)^{|C|}\,\widehat{U_Ci_A}\,.$$

Every subset C determines uniquely a subset $S=C{-}D$ contained in $P{-}D$, so the inner sum can be written

$$\sum_{C\subseteq P}(-1)^{|C|}\,\widehat{U_Ci_A}=\sum_{\substack{S\\ S\subseteq P-D}}\ \sum_{\substack{C\\ S\subseteq C\subseteq S\cup D}}(-1)^{|C|}\,\widehat{U_Ci_A}\,.$$

Since all subsets C in the second sum satisfy $S=C{-}D$, we have $U_Ci_A=U_Si_A$ by Lemma 5.9. Thus all functions U_Ci_A are equal when C runs over all subsets of $S\cup D$ containing S and therefore

$$\sum_{S\subseteq C\subseteq S\cup D}(-1)^{|C|}\,\widehat{U_Ci_A}=\Big(\sum_{S\subseteq C\subseteq S\cup D}(-1)^{|C|}\Big)\,\widehat{U_Si_A}\,.$$

Note that the latter sum has $2^{|D|}$ terms and is zero if $|D|\geq 1$, that is, if $D\neq\varnothing$. If A is a proper subset of P, then $P{-}A\neq\varnothing$, hence $D\neq\varnothing$, and it follows that

$$\sum_{C\subseteq P}(-1)^{|C|}\,\widehat{U_Ci_A}=\sum_{\substack{S\\ S\subseteq P-D}}\ \sum_{\substack{C\\ S\subseteq C\subseteq S\cup D}}(-1)^{|C|}\,\widehat{U_Ci_A}=\sum_{\substack{S\\ S\subseteq P-D}}0\;=0\,.$$

Thus the sum over A at the beginning of the proof reduces to a single term for $A=P$ and we are left with $(-1)^{|P|}\sum_{C\subseteq P}(-1)^{|C|}\,\widehat{U_Ci_P}$, as was to be shown. □

Now we come to the last step in our series of reductions.

5.11. Lemma. *Let $\theta: G^E\to T$ be the inclusion map. For every subset C of P, we have $U_Ci_P=\theta_{\overline{C}}$, where $\overline{C}=P{-}C$.*

Proof : We apply Lemma 5.6 in the special case when $A=P$.

(a) If $g\in E{-}C$, then $(U_Ci_P)(g)=\begin{cases} g & \text{if } g\notin P\,,\\ s_T(g) & \text{if } g\in P\,.\end{cases}$

(b) If $g\in C$ and $g\notin P^\infty$, then $\rho(g)\in P$ and so $(U_Ci_P)(g)=g$.

(c) If $g\in C$ and $g\in P^\infty$, then $(U_Ci_P)(g)=g$.

(d) If $g\in\bigvee E{-}E$, then $(U_Ci_P)(g)=g$.

(e) If $g\in\widetilde{G}^E$, then $\rho(g)\in P$, hence $(U_Ci_P)(g)=g$.

Notice that $E{-}C=(E{-}P)\sqcup\overline{C}$. It follows that

$$(U_Ci_P)(g)=\begin{cases} g & \text{if } g\notin\overline{C}\,,\\ s_T(g) & \text{if } g\in\overline{C}\,.\end{cases}$$

These are precisely the values of the function $\theta_{\overline{C}}$. □

Proof of Theorem 5.4. Starting from Corollary 5.10 and replacing each $U_C i_P$ by $\theta_{\overline{C}}$ (by Lemma 5.11), we obtain

$$\sum_{C\subseteq P}(-1)^{|C|}\,U_C\,\widehat{i} = (-1)^{|P|}\sum_{C\subseteq P}(-1)^{|C|}\,\widehat{U_C i_P} = (-1)^{|P|}\sum_{C\subseteq P}(-1)^{|C|}\,\widehat{\theta_{\overline{C}}}\,.$$

But $\widehat{\theta_{\overline{C}}} = 0$ if $\overline{C} \neq \varnothing$ by Lemma 3.10. So we are left with the case $\overline{C} = \varnothing$, that is, $C = P$, and we get

$$\sum_{C\subseteq P}(-1)^{|C|}\,U_C\,\widehat{i} = (-1)^{|P|}(-1)^{|P|}\,\widehat{\theta_\varnothing} = \widehat{\theta}\,,$$

as was to be shown. ∎

5.12. Corollary. *The subfunctor of Q_T^E generated by $\widehat{\theta}$ is equal to the subfunctor S_T^E generated by $\widehat{i}$.*

Proof : Let $U = \{(e,e) \mid e \in E\} \subseteq E \times G^E$. It is trivial to check that, for any $A \subseteq E$, we have $U\theta_A = i_A$. Therefore $U\widehat{\theta} = \widehat{i}$, so that S_T^E is contained in the subfunctor of Q_T^E generated by $\widehat{\theta}$. The reverse inclusion follows from the fact that $\widehat{\theta}$ belongs to $S_T^E(G^E)$, by Theorem 5.4. ∎

6. A basis for each evaluation

Having achieved the largest part of the work with the universal element $\widehat{\theta}$ in Section 5, it is now easy to prove our main result about the evaluation $S_T^E(X)$ at some finite set X. We need to prove that $\widehat{\varphi} \in S_T^E(X)$ for any function $\varphi \in T^X$ such that $E \subseteq \varphi(X) \subseteq G^E$, where we write again $\widehat{\varphi} = \widehat{\varphi}^E$ for simplicity of notation. We first start with two lemmas.

6.1. Lemma. *Let $\eta \in T^Y$ and let $\psi : X \to Y$ be a map. Then $\widehat{\eta\circ\psi} = \widehat{\eta}\circ\psi$.*

Proof : For any subset $A \subseteq E$, we have

$$(\eta\circ\psi)_A = s_T^A \circ(\eta\circ\psi) = (s_T^A\circ\eta)\circ\psi = \eta_A\circ\psi\,.$$

Taking the alternating sum over A, we get $\widehat{\eta\circ\psi} = \widehat{\eta}\circ\psi$. ∎

6.2. Lemma. *Let η be a k-linear combination of maps in T^Y. Let $\psi : X \to Y$ be a map and let V_ψ be the correspondence $V_\psi = \{(x,\psi(x)) \mid x \in X\} \subseteq X \times Y$. Then $\eta\circ\psi = V_\psi\eta$.*

Proof : By linearity, it suffices to prove the result when η is a map in T^Y. We have $(V_\psi\eta)(x) = \bigvee\limits_{(x,y)\in V_\psi}\eta(y)$. But the join has a single term for $y = \psi(x)$, by the definition of V_ψ. Therefore $(V_\psi\eta)(x) = \eta(\psi(x))$. ∎

6.3. Theorem. *For any finite set X, let $\mathcal{B}^E_X = \{\varphi \in T^X \mid E \subseteq \varphi(X) \subseteq G^E\}$ and $\widehat{\mathcal{B}}^E_X = \{\widehat{\varphi} \mid \varphi \in \mathcal{B}^E_X\}$.*

(a) *For any $\varphi \in \mathcal{B}^E_X$,*

$$\widehat{\varphi} = \sum_{C \subseteq P} (-1)^{|C|} \, (V_\varphi U_C) \, \widehat{i} \,,$$

where $V_\varphi = \{(x, \varphi(x)) \mid x \in X\} \subseteq X \times G^E$ and where P and U_C are defined by (5.1) and (5.3). In particular, $\widehat{\varphi} \in S^E_T(X)$.

(b) *$\widehat{\mathcal{B}}^E_X$ is a k-basis of $S^E_T(X)$. In other words $S^E_T(X) = k\widehat{\mathcal{B}}^E_X$.*

(c) *The k-module $S^E_T(X)$ is free of rank*

$$\mathrm{rk}_k\big(S^E_T(X)\big) = |\mathcal{B}^E_X| = \sum_{i=0}^{|E|} (-1)^i \binom{|E|}{i} (|G^E| - i)^{|X|} \,.$$

Proof : (a) Since $\varphi(X) \subseteq G^E$, we can write $\varphi = \theta \circ \psi$ where $\theta : G^E \to T$ is the inclusion map and $\psi : X \to G^E$ denotes the map φ with its codomain restricted to G^E. By Lemma 6.1, we have

$$\widehat{\varphi} = \widehat{\theta \circ \psi} = \widehat{\theta} \circ \psi \,.$$

By Theorem 5.4, we know that $\widehat{\theta}$ belongs to $S_T(X)$ with an explicit expression, namely

$$\widehat{\varphi} = \widehat{\theta} \circ \psi = \sum_{C \subseteq P} (-1)^{|C|} \, (U_C \, \widehat{i}) \circ \psi \,.$$

Now $(U_C \, \widehat{i}) \circ \psi = V_\psi (U_C \, \widehat{i})$ by Lemma 6.2, where $V_\psi = V_\varphi = \{(x, \varphi(x)) \mid x \in X\}$. Therefore

$$\widehat{\varphi} = \sum_{C \subseteq P} (-1)^{|C|} \, V_\varphi (U_C \, \widehat{i}) = \sum_{C \subseteq P} (-1)^{|C|} \, (V_\varphi U_C) \, \widehat{i} \,.$$

This shows that $\widehat{\varphi}$ belongs to the subfunctor of F_T generated by $\widehat{i}$. In other words, $\widehat{\varphi} \in S_T(X)$.

(b) Recall that $\widehat{\mathcal{B}}^E_X$ is linearly independent because it is a subset of the basis $\widehat{\mathcal{M}}^E_X$ of $Q^E_T(X)$. Moreover, we know that $S^E_T(X) \subseteq k\widehat{\mathcal{B}}^E_X$ by Theorem 4.6. By (a), $\widehat{\varphi} \in S^E_T(X)$ for any $\varphi \in \mathcal{B}^E_X$, so $k\widehat{\mathcal{B}}^E_X \subseteq S^E_T(X)$.

(c) The formula follows from Lemma 6.5 in [BT4]. ▯

7. Minimal nonzero evaluation

We continue to work with a finite lattice T and a fixed subset $E \subseteq \mathrm{Mirr}(T)$. By Corollary 3.16, we know that $Q^E_T(X) = \{0\}$ if $|X| < |E|$ and that $Q^E_T(E)$ has a k-basis $\{\,\widehat{i\sigma} \mid \sigma \in \Sigma_E\}$, where $i : E \to T$ is the inclusion map. We first note that the same holds for the subfunctor S^E_T.

7.1. Lemma. *$S^E_T(X) = \{0\}$ if $|X| < |E|$. Moreover, $S^E_T(E)$ has a k-basis*

$$\{\,\widehat{i\sigma} \mid \sigma \in \Sigma_E\} = \{\Delta_\tau \, \widehat{i} \mid \tau \in \Sigma_E\} \,,$$

where $\Delta_\tau = \{(\tau(e), e) \mid e \in E\}$. In particular, $S^E_T(E) = Q^E_T(E)$.

Proof : Since $Q^E_T(X) = \{0\}$ if $|X| < |E|$, it is clear that $S^E_T(X) = \{0\}$ if $|X| < |E|$. Let $\widehat{i\sigma}$ be a basis element of $Q^E_T(E)$. Lemmas 6.1 and 6.2 show that

$$\widehat{i\sigma} = \widehat{i}\sigma = U_\sigma \, \widehat{i} = \Delta_{\sigma^{-1}} \, \widehat{i} \,,$$

where $U_\sigma = \Delta_{\sigma^{-1}} = \{(e,\sigma(e)) \mid e \in E\}$. Since each $\Delta_{\sigma^{-1}}\,\widehat{i}$ belongs to $S_T^E(E)$, we get a k-basis of $S_T^E(E)$ and $S_T^E(E) = Q_T^E(E)$. $\square$

We aim to describe the minimal nonzero evaluation $S_T^E(E)$, viewed as a left $k\mathcal{R}_E$-module, where $\mathcal{R}_E$ denotes the monoid of all relations on E (i.e. correspondences in $E\times E$) and $k\mathcal{R}_E$ is the monoid algebra of $\mathcal{R}_E$ (where k is a commutative ring).

Associated to a poset (E,R) (i.e. R is a partial order on the set E) there is a specific $k\mathcal{R}_E$-module $M_{E,R}$ which we call the *fundamental* $k\mathcal{R}_E$-module and which plays a major role in all our work on correspondence functors. The module $M_{E,R}$ appears in Section 7 of [BT1] as a left ideal $\mathcal{P}f_R$, where f_R is a suitable idempotent in some quotient algebra $\mathcal{P}$ of $k\mathcal{R}_E$. The action (7.2) below appears in Proposition 8.5 of [BT1].

We define $M_{E,R}$ to be the free k-module with a basis $\{m_\sigma \mid \sigma\in\Sigma_E\}$, where Σ_E is the group of all permutations of the set E. We let $\Delta_\tau = \{(\tau(e),e)\mid e\in E\}\subseteq E\times E$ and ${}^\sigma R = \Delta_\sigma R\Delta_{\sigma^{-1}}$, while $\Delta=\Delta_{\mathrm{id}}$ is the diagonal subset of $E\times E$. Note that $\Delta_\sigma\Delta_\tau = \Delta_{\sigma\tau}$ and that Δ_σ is invertible with inverse $\Delta_{\sigma^{-1}}$. The left $k\mathcal{R}_E$-module structure is entirely determined by specifying the action of each relation $U\in\mathcal{R}_E$ on the k-basis :

$$\textbf{(7.2)}\qquad U\cdot m_\sigma = \begin{cases} m_{\tau\sigma} & \text{if } \exists\,\tau\in\Sigma_E \text{ such that } \Delta\subseteq\Delta_{\tau^{-1}}U\subseteq {}^\sigma R\,,\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, if τ exists with this condition, then it is unique. Notice that $M_{E,R}$ is generated by $m := m_{\mathrm{id}}$ and moreover $\Delta_\tau\cdot m = m_\tau$.

7.3. Lemma. *Let (E,R) be a finite poset and let M be a left $k\mathcal{R}_E$-module such that:*

(a) M is a free k-module with a basis $\{\Delta_\sigma\cdot m\mid\sigma\in\Sigma_E\}$ for some $m\in M$,
(b) $U\cdot m=0$ if $U\in\mathcal{R}_E$ does not contain any permutation,
(c) $U\cdot m=m$ if $U\in\mathcal{R}_E$ is reflexive and $U\subseteq R$,
(d) $U\cdot m=0$ if $U\in\mathcal{R}_E$ is reflexive and $U\nsubseteq R$.

Then M is isomorphic to the fundamental module $M_{E,R}$, generated by m.

Proof : Let $m_\sigma=\Delta_\sigma\cdot m$ and let $U\in\mathcal{R}_E$. Assume first that U does not contain any permutation. Then the inclusion $\Delta\subseteq\Delta_{\tau^{-1}}U$ cannot hold. On the other hand, $U\Delta_\sigma$ does not contain any permutation, so $U\Delta_\sigma\cdot m=0$ by (b), that is, $U\cdot m_\sigma=0$.

Assume now that U contains a permutation Δ_τ. Then $\Delta_{\tau^{-1}}U$ is reflexive, that is, $\Delta\subseteq\Delta_{\tau^{-1}}U$. We fix $\sigma\in\Sigma_E$ and we need to compute $U\cdot m_\sigma$.

Suppose that $\Delta_{\tau^{-1}}U\subseteq{}^\sigma R$. Conjugating on the right by Δ_σ, we get

$$\Delta\subseteq\Delta_{\sigma^{-1}}\Delta_{\tau^{-1}}U\Delta_\sigma\subseteq R\,.$$

By (c) applied to the reflexive relation $\Delta_{\sigma^{-1}}\Delta_{\tau^{-1}}U\Delta_\sigma$, we obtain

$$\Delta_{\sigma^{-1}}\Delta_{\tau^{-1}}U\Delta_\sigma\cdot m = m\,.$$

Multiplying on the left by $\Delta_{\tau\sigma}$, we have $U\Delta_\sigma\cdot m=\Delta_{\tau\sigma}\cdot m$, that is, $U\cdot m_\sigma=m_{\tau\sigma}$.

Suppose now that $\Delta_{\tau^{-1}}U\nsubseteq{}^\sigma R$. Then

$$\Delta\subseteq\Delta_{\sigma^{-1}}\Delta_{\tau^{-1}}U\Delta_\sigma\nsubseteq R\,.$$

By (d), we have $\Delta_{\sigma^{-1}}\Delta_{\tau^{-1}}U\Delta_\sigma\cdot m=0$. Multiplying on the left by $\Delta_{\tau\sigma}$, we obtain $U\Delta_\sigma\cdot m=0$, that is, $U\cdot m_\sigma=0$.

This completes the proof that the defining properties (7.2) hold. $\square$

Since E is a subset of the lattice T, it is naturally partially ordered and we write

$$R=\{(e,f)\in E\times E\mid e\leq f\}\,.$$

It is actually the opposite partial order R^{op} which comes into play.

7.4. Theorem. *Let $E \subseteq \mathrm{Mirr}(T)$ and let R^{op} be the opposite order of the natural partial order on E. The $k\mathcal{R}_E$-module $S_T^E(E)$, generated by $\widehat{i}$, is isomorphic to the fundamental module $M_{E,R^{op}}$, generated by m.*

Proof : We will apply Lemma 7.3 to the $\mathcal{R}_E$-module $S_T^E(E)$, its specific element $\widehat{i}$, and the partial order R^{op}. By Lemma 7.1, we know that $S_T^E(E)$ has a k-basis

$$\{\Delta_\sigma \widehat{i} \mid \sigma \in \Sigma_E\}\,,$$

so condition (a) in Lemma 7.3 holds.

Let $U \in \mathcal{R}_E$ be a relation. Suppose first that U does not contain any permutation. Then Theorem 3.2 in [BT1] asserts that U is inessential, that is, U factors through a set X of cardinality strictly smaller than $|E|$. In other words, $U = VW$ where $V \subseteq E \times X$ and $W \subseteq X \times E$. It follows that $U\widehat{i} = VW\widehat{i} = 0$, because $W\widehat{i} \in S_T^E(X) = \{0\}$ by Lemma 7.1.

Suppose next that U is reflexive and $U \subseteq R^{op}$. Then, for any subset A of E and any $f \in E$, we have

$$(Ui_A)(f) = \bigvee_{(f,e)\in U} i_A(e)\,.$$

Since $U \subseteq R^{op}$, the condition $(f,e) \in U$ implies that $(f,e) \in R^{op}$, that is, $e \leq f$, hence $i_A(e) \leq i_A(f)$. On the other hand, $(f,f) \in U$ by reflexivity of U. It follows that the join above is equal to the single term $i_A(f)$ and so $(Ui_A)(f) = i_A(f)$. This shows that $Ui_A = i_A$, hence $U\widehat{i} = \widehat{i}$.

Suppose finally that U is reflexive and $U \not\subseteq R^{op}$. The reflexivity of U implies that, for any subset A of E,

$$(Ui_A)(f) = \bigvee_{(f,e)\in U} i_A(e) \geq i_A(f) \geq f\,,$$

hence $Ui_A \geq i$. We claim that if a function $\varphi : E \to T$ satisfies $\varphi > i$ (i.e. $\varphi(x) \geq x$ for every $x \in E$ and $\varphi(e) > e$ for some $e \in E$), then $\widehat{\varphi} = 0$.

Postponing the proof of this claim, we apply it to the function Ui_A. If $A \neq \varnothing$, then $Ui_A \geq i_A > i$, because for any $e \in A$ we have $i_A(e) = s_T(e) > e$. Thus $\widehat{Ui_A} = 0$. If $A = \varnothing$, then $i_A = i$. Since $U \not\subseteq R^{op}$, there exists $(f,g) \in U$ such that $(f,g) \notin R^{op}$, that is, $g \not\leq f$. Then

$$(Ui)(f) = \bigvee_{(f,e)\in U} i(e) \geq i(f) \vee i(g) = f \vee g > f$$

because $g \not\leq f$. It follows that $Ui > i$, hence $\widehat{Ui} = 0$. We have now proved that $\widehat{Ui_A} = 0$ for all A and, by Lemma 4.1, this implies that

$$U\widehat{i} = \sum_{A\subseteq E} (-1)^{|A|}\,\widehat{Ui_A} = 0\,.$$

We see that all the assumptions of Lemma 7.3 are satisfied, so that $S_T^E(E)$ is isomorphic to the fundamental module $M_{E,R^{op}}$.

We are left with the proof of the claim. Let $\varphi \in T^E$ such that $\varphi > i$. Let $e \in E$ be minimal such that $\varphi(e) \neq e$. If we had $e \in \varphi(E)$, then we would have $e = \varphi(f)$ for some $f \in E$. We must have $f \neq e$ because $\varphi(f) \neq \varphi(e)$. Moreover, $e = \varphi(f) \geq i(f) = f$, hence $e > f$. By minimality of e, we deduce that $\varphi(f) = f$, hence $e = f$, a contradiction. It follows that $e \notin \varphi(E)$, so that $\varphi \notin \mathcal{M}_E$ and $\widehat{\varphi} = 0$ by Lemma 3.10. ∎

8. An embedding property

In this section, we show that the functors Q_T and S_T have an important injectivity property which will be used repeatedly, in particular as the main ingredient in our next section on fundamental functors. In order to obtain the results, we cannot use an arbitrary subset E of $\mathrm{Mirr}(T)$, so we assume from now on that $E = \mathrm{Mirr}(T)$. Consequently, we ignore the superscript E and in particular we have $Q_T = Q_T^E$ and $S_T = S_T^E$.

Let X be a finite set. Any correspondence $W \in \mathcal{C}(E, X)$ induces a k-linear map

$$Q_T(X) \longrightarrow Q_T(E)\,, \qquad \omega \mapsto W\omega\,.$$

The product of all these maps when W runs over correspondences in $E \times X$ yields a map

$$\mu_X : Q_T(X) \longrightarrow \prod_{W \in \mathcal{C}(E,X)} Q_T(E)\,, \qquad \omega \mapsto \big(W\omega\big)_{W \in \mathcal{C}(E,X)}\,.$$

8.1. Theorem. *The map μ_X above is injective. Explicitly, if $\omega \in Q_T(X)$ satisfies $W\omega = 0$ for any correspondence $W \in E \times X$, then $\omega = 0$.*

Proof : By Theorem 3.12, we can write

$$\omega = \sum_{\varphi \in \mathcal{M}_X} \lambda_\varphi \widehat{\varphi}\,,$$

where $\lambda_\varphi \in k$, $\mathcal{M}_X = \mathcal{M}_X^E$ and $\widehat{\varphi} = \widehat{\varphi}^E$. Therefore $0 = W\omega = \sum_{\varphi \in \mathcal{M}_X} \lambda_\varphi W\widehat{\varphi}$ for any correspondence $W \subseteq E \times X$ and we now make a specific choice. For any function $\pi \in \mathcal{M}_X$, we define

$$W_\pi = \{(e, x) \in E \times X \mid e \geq \pi(x)\}\,.$$

It follows that

$$0 = \widehat{W_\pi\,\omega} = \sum_{\varphi \in \mathcal{M}_X} \lambda_\varphi \widehat{W_\pi\,\widehat{\varphi}}\,. \tag{8.2}$$

Now $\widehat{\varphi} = \sum_{A \subseteq E} (-1)^{|A|} \varphi_A$ and, by Lemma 3.10, we have $W_\pi\,\varphi_A \in \mathcal{M}_X$ whenever $\widehat{W_\pi\,\varphi_A} \neq 0$. The definition of $\mathcal{M}_X$ yields $E \subseteq W_\pi\,\varphi_A(E)$ and therefore $W_\pi\,\varphi_A = i \circ \sigma_A$ for some permutation $\sigma_A \in \Sigma_E$, where $i : E \to T$ denotes the inclusion map. By Lemma 7.1 and its proof, we get

$$\widehat{W_\pi\,\varphi_A} = \widehat{i\,\sigma_A} = \Delta_{\sigma_A^{-1}}\,\widehat{i}\,,$$

and therefore the equation (8.2) above becomes

$$\begin{aligned} 0 &= \sum_{\varphi \in \mathcal{M}_X} \lambda_\varphi \sum_{\sigma \in \Sigma_E} \Big(\sum_{\substack{A \subseteq E \\ \sigma_A = \sigma}} (-1)^{|A|}\, \widehat{i\,\sigma_A} \Big) \\ &= \sum_{\sigma \in \Sigma_E} \Big(\sum_{\varphi \in \mathcal{M}_X} \lambda_\varphi \sum_{\substack{A \subseteq E \\ W_\pi\,\varphi_A = i\sigma}} (-1)^{|A|} \Big) \Delta_{\sigma^{-1}}\,\widehat{i}\,. \end{aligned}$$

because $\sigma_A = \sigma$ if and only if $W_\pi\,\varphi_A = i\sigma$. Since $\{\Delta_\tau\,\widehat{i} \mid \tau \in \Sigma_E\}$ is a k-basis of $Q_T(E)$ (by Lemma 7.1), the coefficients vanish. In particular, when $\sigma = \mathrm{id}$, we get

$$\sum_{\varphi \in \mathcal{M}_X} \lambda_\varphi \sum_{\substack{A \subseteq E \\ W_\pi\,\varphi_A = i}} (-1)^{|A|} = 0\,.$$

Defining a square matrix (indexed by $\mathcal{M}_X \times \mathcal{M}_X$) by

$$\varepsilon_{\pi,\varphi} = \sum_{\substack{A\subseteq E \\ W_\pi\,\varphi_A = i}} (-1)^{|A|}\,,$$

we obtain

$$\sum_{\varphi\in\mathcal{M}_X} \varepsilon_{\pi,\varphi}\,\lambda_\varphi = 0\,.$$

In other words, the product of the matrix with the column vector (λ_φ) is equal to zero.

The condition $W_\pi\,\varphi_A = i$ gives, for any $f \in E$,

$$f = i(f) = W_\pi\,\varphi_A(f) = \bigvee_{\substack{y\in X \\ (f,y)\in W_\pi}} \varphi_A(y) = \bigvee_{\substack{y\in X \\ f\geq \pi(y)}} \varphi_A(y) \geq \bigvee_{\substack{y\in X \\ f\geq \pi(y)}} \varphi(y) \geq \varphi(x)$$

for any $x \in X$ such that $f \geq \pi(x)$. Since every element of T is a meet of meet-irreducible elements, see (2.3), we obtain

$$\pi(x) = \bigwedge_{\substack{f\in E \\ f\geq \pi(x)}} f \geq \varphi(x)\,,$$

showing that $\pi \geq \varphi$. Therefore, if $\pi \not\geq \varphi$, the condition $W_\pi\,\varphi_A = i$ cannot hold, $\varepsilon_{\pi,\varphi} = 0$, and so the matrix $(\varepsilon_{\pi,\varphi})$ is triangular.

In order to find the diagonal entries of the matrix, let $\pi = \varphi$ and assume that the condition $W_\varphi\,\varphi_A = i$ holds. Then

$$\textbf{(8.3)} \qquad f = \bigvee_{\substack{y\in X \\ f\geq \varphi(y)}} \varphi_A(y) = \Big(\bigvee_{\substack{y\in X \\ f\geq\varphi(y) \\ \varphi(y)\notin A}} \varphi(y)\Big) \vee \Big(\bigvee_{\substack{z\in X \\ f\geq\varphi(z) \\ \varphi(z)\in A}} s_T(\varphi(z))\Big)\,,$$

because $\varphi_A(z) = s_T(\varphi(z))$ when $\varphi(z) \in A$. Suppose that $A \neq \emptyset$ and let $f \in A$. Since $\varphi \in \mathcal{M}_X$, we have $E \subseteq \varphi(X)$, so there exists $z_0 \in X$ such that $f = \varphi(z_0)$. The equation (8.3) yields in particular $f \geq s_T(\varphi(z_0))$, hence

$$f \geq s_T(\varphi(z_0)) > \varphi(z_0) = f\,,$$

a contradiction. Therefore A must be empty, $\varphi_\varnothing = \varphi$, and we get

$$\varepsilon_{\varphi,\varphi} = \sum_{\substack{A\subseteq E \\ W_\varphi\,\varphi_A = i}} (-1)^{|A|} = \begin{cases} 1 & \text{if } W_\varphi\,\varphi = i\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

But since $E \subseteq \varphi(X)$, for any $f \in E$, there exists $y_f \in X$ such that $f = \varphi(y_f)$. Therefore

$$W_\varphi\,\varphi(f) = \bigvee_{\substack{y\in X \\ f\geq\varphi(y)}} \varphi(y) = \varphi(y_f) = f = i(f)\,.$$

This shows that the condition $W_\varphi\,\varphi = i$ holds and so the diagonal coefficient $\varepsilon_{\varphi,\varphi}$ is equal to 1. Therefore, the matrix is unitriangular, hence invertible. By inverting the matrix, we deduce that the column vector (λ_φ) is equal to zero. Therefore $\omega = \sum_{\varphi\in\mathcal{M}_X} \lambda_\varphi\widehat{\varphi} = 0$. This completes the proof of Theorem 8.1. $\square$

8.4. Corollary. *The map*

$$\mu_X : S_T(X) \longrightarrow \prod_{W\in\mathcal{C}(E,X)} S_T(E)\,, \qquad \omega \mapsto \big(W\omega\big)_{W\in\mathcal{C}(E,X)}\,.$$

is injective.

Proof : Since $S_T(E) = Q_T(E)$ by Lemma 7.1, the map μ_X is the restriction to $S_T(X)$ of the map μ_X of Theorem 8.1. ∎

8.5. Remark. We can interpret Theorem 8.1 in a more functorial way. Evaluation at E is a functor from the category of correspondence functors to the category of (left) $k\mathcal{R}_E$-modules. This has a left adjoint, mapping a $k\mathcal{R}_E$-module V to the correspondence functor

$$L_{E,V} := k\mathcal{C}(-,E)\otimes_{k\mathcal{R}(E)} V\,,$$

as well as a right adjoint, mapping V to the correspondence functor

$$L^0_{E,V} := \mathrm{Hom}_{k\mathcal{R}(E)}\big(k\mathcal{C}(E,-),V\big)\,.$$

For any correspondence functor F, the unit of the latter adjunction yields a morphism of correspondence functors

$$F \longrightarrow L^0_{E,F(E)}\,.$$

Taking $F = Q_T$ and evaluating at a finite set X, we get a map

$$Q_T(X) \longrightarrow L^0_{E,Q_T(E)}(X) = \mathrm{Hom}_{k\mathcal{R}(E)}\big(k\mathcal{C}(E,X),Q_T(E)\big) \subseteq \prod_{W\in\mathcal{C}(E,X)} Q_T(E)$$

which is equal to the injective map μ_X of Theorem 8.1. Therefore, this theorem asserts in fact that the correspondence functor Q_T is isomorphic to a subfunctor of $L^0_{E,Q_T(E)}$, which is a quite explicit functor in view of the description of $Q_T(E)$ in Corollary 3.16.

9. Fundamental functors

Fundamental functors play an important role in our work and are in particular essential in [BT4]. In this section, our main purpose is to prove that the functor $S_T := S_T^E$ of the present paper is isomorphic to the fundamental functor $S_{E,M}$ where $M = M_{E,R^{op}}$ is the fundamental module defined in Section 7. Here $E = \mathrm{Mirr}(T)$ as in Section 8.

We need to define the notion of fundamental functor. Recall from Section 2 in [BT3] that, for any left $k\mathcal{R}_E$-module V, there is an associated correspondence functor

$$L_{E,V} := k\mathcal{C}(-,E)\otimes_{k\mathcal{R}(E)} V\,,$$

(see Remark 8.5 above), a subfunctor $J_{E,V}$ defined by

$$J_{E,V}(X) := \Big\{\sum_i \alpha_i\otimes w_i \in L_{E,V}(X) \mid \forall\beta\in k\mathcal{C}(E,X), \sum_i(\beta\alpha_i)\cdot w_i = 0\Big\}$$

and a quotient

$$S_{E,V} := L_{E,V}/J_{E,V}\,.$$

When E is equal to $\mathrm{Mirr}(T)$ (for some finite lattice T) and $M = M_{E,R^{op}}$ is the fundamental $k\mathcal{R}_E$-module, we define

$$\mathbb{S}_{E,R} := S_{E,M}$$

and call it the *fundamental correspondence functor* associated to the finite poset (E,R). Here, and for the rest of this paper, we emphasize that a finite poset is a pair (E,R) where E is a finite set and R is a partial order relation on the set E.

9.1. Theorem. *Let T be a finite lattice and $M = M_{E,R^{op}}$ be the fundamental $k\mathcal{R}_E$-module, where $E = \mathrm{Mirr}(T)$ and R^{op} is the opposite order of the natural partial order on E. Then S_T is isomorphic to the fundamental correspondence functor $\mathbb{S}_{E,R}$.*

Proof : The isomorphism $M \cong S_T(E)$ of Theorem 7.4 extends to a morphism of functors

$$\eta : L_{E,M} \longrightarrow S_T \, .$$

This is surjective because S_T is generated by $\widehat{i}$, which belongs to $S_T(E)$. The definition of the subfunctor $J_{E,M}$ shows that the image of $J_{E,M}$ under η is equal to the subfunctor K of S_T defined by

$$K(X) = \{\omega \in S_T(X) \mid W\omega = 0 \text{ for all } W \in \mathcal{C}(E,X)\} \, .$$

By Corollary 8.4, we have $K(X) = \{0\}$. Therefore $K = 0$ and $J_{E,M}$ is the kernel of η. It follows that

$$\mathbb{S}_{E,R} = S_{E,M} = L_{E,M}/J_{E,M} \cong S_T \, ,$$

as required. □

By using Theorem 6.3, we recover in particular a k-basis for $\mathbb{S}_{E,R}(X)$ and a formula for its rank, providing a new proof of Theorem 6.6 in [BT4].

10. Simple functors

Simple correspondence functors are classified and studied in [BT2, BT3, BT4]. Our purpose in this section is to show that they can be constructed by using in a direct fashion the functor S_T of the present paper, independently of our previous work. In order to describe this construction, we first show that the group $A = \mathrm{Aut}(E,R)$ acts on the right on the functor S_T. As usual, T is a finite lattice, $\mathrm{Mirr}(T) = (E,R)$ and we let $A = \mathrm{Aut}(E,R)$.

For any finite set X and any correspondence $U \in \mathcal{C}(X,E)$, we define $\alpha_X(U) = U\,\widehat{i}$ and we extend this by k-linearity to a k-linear map

$$\alpha_X : k\mathcal{C}(X,E) \longrightarrow S_T(X) \, ,$$

which is surjective by the definition of S_T. This clearly defines a morphism of correspondence functors $\alpha : k\mathcal{C}(-,E) \longrightarrow S_T$. The group algebra kA has a right permutation action on $k\mathcal{C}(X,E)$ defined by $U \cdot \gamma = U\Delta_\gamma$ for any $U \in \mathcal{C}(X,E)$ and any $\gamma \in A$. As before, we write $\Delta_\tau = \{(\tau(e),e) \mid e \in E\}$ for any permutation τ of E. We intend to show that this action passes to the quotient $k\mathcal{C}(X,E)/\mathrm{Ker}(\alpha_X) \cong S_T(X)$. Explicitly, we will show that $\mathrm{Ker}(\alpha_X)$ is a kA-submodule of $k\mathcal{C}(X,E)$, so that the right kA-module structure can be transported from $k\mathcal{C}(X,E)$ to $S_T(X)$ via the map α_X. We first prove this in the case $X = E$.

10.1. Lemma. $\mathrm{Ker}(\alpha_E)$ *is a kA-submodule of $k\mathcal{C}(E,E)$. Consequently, the action*

$$(U\widehat{i})\cdot\gamma := (U\Delta_\gamma)\widehat{i}\,, \qquad \forall\, U\in\mathcal{C}(E,E)\,,\forall\,\gamma\in A\,,$$

is a well-defined right kA-module structure on $S_T(E)$.

Proof : Let $\omega = \sum_{U\in\mathcal{C}(E,E)} \lambda_U U \in k\mathcal{C}(E,E)$, where $\lambda_U \in k$, and suppose that $\omega\in\mathrm{Ker}(\alpha_E)$, that is,

$$\sum_{U\in\mathcal{C}(E,E)} \lambda_U U\widehat{i} = 0\,.$$

This belongs to $S_T(E)\cong M_{E,R^{op}}$ (Theorem 7.4) and the definition (7.2) yields

$$U\widehat{i} = \begin{cases} \Delta_\tau\widehat{i} & \text{if } \exists\,\tau\in\Sigma_E \text{ such that } \Delta\subseteq\Delta_{\tau^{-1}}U\subseteq R^{op}\,,\\ 0 & \text{otherwise,}\end{cases}$$

with τ unique if it exists. This uniqueness allows us to sum over all permutations τ, under the condition $\Delta\subseteq\Delta_{\tau^{-1}}U\subseteq R^{op}$. Explicitly, we get

$$U\widehat{i} = \sum_{\substack{\tau\in\Sigma_E\\ \Delta\subseteq\Delta_{\tau^{-1}}U\subseteq R}} \Delta_\tau\widehat{i}\,,$$

where all terms of the sum are zero, except at most one. It follows that

$$\begin{aligned} 0 = \sum_{U\in\mathcal{C}(E,E)} \lambda_U U\widehat{i} &= \sum_{U\in\mathcal{C}(E,E)} \lambda_U \sum_{\substack{\tau\in\Sigma_E\\ \Delta\subseteq\Delta_{\tau^{-1}}U\subseteq R^{op}}} \Delta_\tau\widehat{i}\\ &= \sum_{\tau\in\Sigma_E}\Big(\sum_{\substack{U\in\mathcal{C}(E,E)\\ \Delta\subseteq\Delta_{\tau^{-1}}U\subseteq R^{op}}} \lambda_U\Big)\Delta_\tau\widehat{i}\,.\end{aligned}$$

Since $\{\Delta_\tau\widehat{i}\mid\tau\in\Sigma_E\}$ is a k-basis of $S_T(E)$ (Lemma 7.1), the coefficients must vanish. Thus, for any $\tau\in\Sigma_E$, we get

$$\sum_{\substack{U\in\mathcal{C}(E,E)\\ \Delta\subseteq\Delta_{\tau^{-1}}U\subseteq R^{op}}} \lambda_U = 0\,. \tag{10.2}$$

Now we have to prove that $\omega\Delta_\gamma$ belongs to $\mathrm{Ker}(\alpha_E)$ for any $\gamma\in A$. We have $\omega\Delta_\gamma = \sum_{U\in\mathcal{C}(E,E)}\lambda_U U\Delta_\gamma$ and the definition (7.2) yields

$$U\Delta_\gamma\widehat{i} = \begin{cases} \Delta_{\tau\gamma}\widehat{i} & \text{if } \exists\,\tau\in\Sigma_E \text{ such that } \Delta\subseteq\Delta_{\tau^{-1}}U\subseteq {}^{\gamma}R^{op}\,,\\ 0 & \text{otherwise.}\end{cases}$$

with τ unique if it exists. Here ${}^{\gamma}R^{op} = \Delta_\gamma R^{op}\Delta_{\gamma^{-1}}$ and it easy to check that

$$\gamma\in\mathrm{Aut}(E,R)=\mathrm{Aut}(E,R^{op}) \iff {}^{\gamma}R^{op}=R^{op}\,. \tag{10.3}$$

Therefore the condition above is simply $\Delta\subseteq\Delta_{\tau^{-1}}U\subseteq R^{op}$, as before. We then compute

$$\begin{aligned} \omega\Delta_\gamma\widehat{i} &= \sum_{U\in\mathcal{C}(E,E)} \lambda_U U\Delta_\gamma\widehat{i} = \sum_{U\in\mathcal{C}(E,E)} \lambda_U \sum_{\substack{\tau\in\Sigma_E\\ \Delta\subseteq\Delta_{\tau^{-1}}U\subseteq R^{op}}} \Delta_{\tau\gamma}\widehat{i}\\ &= \sum_{\tau\in\Sigma_E}\Big(\sum_{\substack{U\in\mathcal{C}(E,E)\\ \Delta\subseteq\Delta_{\tau^{-1}}U\subseteq R^{op}}} \lambda_U\Big)\Delta_{\tau\gamma}\widehat{i}\,.\end{aligned}$$

The inner sum is zero, thanks to the equation (10.2) above. Therefore

$$\alpha_E(\omega\Delta_\gamma) = \omega\Delta_\gamma\widehat{i} = 0\,,$$

as was to be shown. ∎

Now we move from E to an arbitrary finite set X. Recall that $G = G^{\mathrm{Mirr}(T)} = \bigvee E \sqcup \widetilde{G}$, as defined in (4.4).

10.4. Lemma.

(a) $\mathrm{Ker}(\alpha_X)$ *is a kA-submodule of $k\mathcal{C}(X,E)$.*

(b) The action

$$(U\,\widehat{i}\,)\cdot\gamma := (U\Delta_\gamma)\,\widehat{i}\,, \qquad \forall\, U\in\mathcal{C}(X,E)\,,\forall\,\gamma\in A\,,$$

is a well-defined right kA-module structure on $S_T(X)$.

(c) Right multiplication by $\gamma\in A$ is a morphism of correspondence functors $S_T\to S_T$.

(d) Every $\gamma\in A$ extends to an automorphism of the full subposet G of T and we have

$$\widehat{\varphi}\cdot\gamma = \widehat{\gamma^{-1}\varphi} \qquad \forall\,\varphi\in G^X \text{ such that } E\subseteq\varphi(X)\,.$$

Proof : (a) Let $\omega\in k\mathcal{C}(X,E)$ such that $\alpha_X(\omega)=0$. We apply the (left) action of a correspondence $W\in\mathcal{C}(E,X)$. Since $W\omega\,\widehat{i}$ belongs to $S_T(E)$, we can use the action of $\gamma\in A$, which is well-defined in $S_T(E)$ by Lemma 10.1. Thus we have

$$0 = (W\alpha_X(\omega))\cdot\gamma = (W\omega\,\widehat{i}\,)\cdot\gamma = W\omega\Delta_\gamma\,\widehat{i} = \alpha_E(W\omega\Delta_\gamma) = W\alpha_X(\omega\Delta_\gamma)\,,$$

using the fact that $\alpha : k\mathcal{C}(-,E)\longrightarrow S_T$ is a morphism of functors. The equality $W\alpha_X(\omega\Delta_\gamma)=0$ holds for any correspondence $W\in\mathcal{C}(E,X)$, so by Corollary 8.4, we get $\alpha_X(\omega\Delta_\gamma)=0$. In other words, $\omega\Delta_\gamma\in\mathrm{Ker}(\alpha_X)$, as was to be shown.

(b) As mentioned at the beginning of this section, the right kA-module structure can be transported from $k\mathcal{C}(X,E)$ to $S_T(X)$ via the map α_X.

(c) Since $\gamma\in A$ acts on the right while correspondences act on the left, it is clear that these actions commute.

(d) Recall that $G=\bigvee E\sqcup\widetilde{G}$. If $t\in\bigvee E$, we define

$$\gamma(t) = \bigvee_{\substack{e\in E\\ e\leq t}} \gamma(e)\,.$$

If $h\in\widetilde{G}$, then Lemma 4.5 shows that $h=s_T(b_1)$ where $b_1=\rho(h)$ and $b_1\in E$. We define $\gamma(h)=s_T(\gamma(b_1))$. Actually, γ maps the whole chain below h in Lemma 4.5 to a chain below $\gamma(h)$ with the same properties. It follows that $\mathrm{Aut}(E,R)=\mathrm{Aut}(G)$, as already observed in Corollary 2.31 of [Bo2]. Recall from Theorem 6.3 that

$$\widehat{\varphi} = \sum_{C\subseteq P}(-1)^{|C|}\,(V_\varphi U_C)\,\widehat{i}\,,$$

where $V_\varphi=\{(x,\varphi(x))\mid x\in X\}\subseteq X\times G$ and where P and U_C are defined by (5.1) and (5.3). We claim that $U_C\Delta_\gamma=\Delta_\gamma U_{\gamma^{-1}(C)}$ and $V_\varphi\Delta_\gamma=V_{\gamma^{-1}\varphi}$. It follows from these claims that

$$\begin{aligned}
\widehat{\varphi}\cdot\gamma &= \sum_{C\subseteq P}(-1)^{|C|}\,(V_\varphi U_C)\,\widehat{i}\cdot\gamma = \sum_{C\subseteq P}(-1)^{|C|}\,(V_\varphi U_C\Delta_\gamma)\,\widehat{i}\\
&= \sum_{C\subseteq P}(-1)^{|C|}\,(V_\varphi\Delta_\gamma U_{\gamma^{-1}(C)})\,\widehat{i} = \sum_{C\subseteq P}(-1)^{|C|}\,(V_{\gamma^{-1}\varphi}U_{\gamma^{-1}(C)})\,\widehat{i}\\
&= \sum_{D\subseteq P}(-1)^{|D|}\,(V_{\gamma^{-1}\varphi}U_D)\,\widehat{i} = \widehat{\gamma^{-1}\varphi}\,,
\end{aligned}$$

as required.

We are left with the proof of the claims. Let $g\in G$ and $e\in E$.

$$\begin{aligned}
(g,e)\in U_C\Delta_\gamma \quad &\iff \quad (g,\gamma(e))\in U_C \iff \begin{cases}\gamma(e)<g & \text{if } g\in (G{-}E)\sqcup C\,,\\ \gamma(e)\le g & \text{if } g\in E{-}C.\end{cases}\\
&\iff \quad \begin{cases} e<\gamma^{-1}(g) & \text{if } \gamma^{-1}(g)\in (G{-}E)\sqcup \gamma^{-1}(C)\,,\\ e\le \gamma^{-1}(g) & \text{if } g\in (E{-}\,\gamma^{-1}(C)).\end{cases}\\
&\iff \quad (\gamma^{-1}(g),e)\in U_{\gamma^{-1}(C)} \iff (g,e)\in \Delta_\gamma U_{\gamma^{-1}(C)}\,.
\end{aligned}$$

proving the first claim. Now let $x\in X$ and $g\in G$.

$$(x,g)\in V_\varphi\Delta_\gamma \iff (\varphi(x),g)\in\Delta_\gamma \iff g=\gamma^{-1}\varphi(x) \iff (x,g)\in V_{\gamma^{-1}\varphi}\,,$$

proving the second claim. ∎

10.5. Lemma. *$S_T(X)$ is a free right kA-module.*

Proof : By Theorem 6.3, $S_T(X)$ has a k-basis $\widehat{\mathcal{B}}_X$ where

$$\mathcal{B}_X=\{\varphi\in T^X \mid E\subseteq\varphi(X)\subseteq G\}=\{\varphi\in G^X \mid E\subseteq \varphi(X)\}\,.$$

Any $\gamma\in A$ permutes $\mathcal{B}_X$ via left composition by γ^{-1}. If $\gamma^{-1}\varphi=\varphi$, then $\gamma^{-1}\varphi(x)=\varphi(x)$ for all $x\in X$, so in particular $\gamma^{-1}(e)=e$ for all $e\in E$ because $E\subseteq\varphi(X)$. Since γ is an automorphism of E, this shows that $\gamma=\mathrm{id}$. Therefore the permutation action of A on $\mathcal{B}_X$ is free.

By Corollary 3.11, the map $\varphi\mapsto\widehat{\varphi}$ is a bijection $\mathcal{B}_X\to\widehat{\mathcal{B}}_X$. Moreover, part (d) of Lemma 10.4 shows that this map also preserves the right action of A. It follows that the permutation action of A on $\widehat{\mathcal{B}}_X$ is free as well. In other words, the kA-module $S_T(X)$ is free. ∎

Let $A=\mathrm{Aut}(E,R)$. For any left kA-module V, we define a correspondence functor $S_{T,V}:=S_T\otimes_{kA}V$, or more precisely

$$S_{T,V}(X):=S_T(X)\otimes_{kA}V\,,$$

using the right action of kA on $S_T(X)$ defined in Lemma 10.4 above. Of course, the (left) action of a correspondence $U\in\mathcal{C}(Y,X)$ on $S_T(X)\otimes_{kA}V$ is given by $U(\omega\otimes v)=(U\omega)\otimes v$, for any $\omega\in S_T(X)$ and any $v\in V$.

In order to treat simple modules and simple functors, we can assume that the base ring k is a field.

10.6. Theorem. *Let k be a field. Let (E,R) be a partial order, $A=\mathrm{Aut}(E,R)$, and T a finite lattice with $\mathrm{Mirr}(T)\cong(E,R)$.*

(a) If V is a simple left kA-module, $S_{T,V}$ is a simple correspondence functor.
(b) If S is a simple correspondence functor, then $S\cong S_{T,V}$ for some finite lattice T with $\mathrm{Mirr}(T)\cong(E,R)$ and some simple kA-module V.
(c) Any simple correspondence functor $S_{T,V}$ is isomorphic to a quotient of a fundamental functor S_T.
(d) The dimension of $S_{T,V}(X)$ is given by the formula

$$\dim_k S_{T,V}(X)=\frac{\dim_k(V)}{|\mathrm{Aut}(E,R)|}\sum_{i=0}^{|E|}(-1)^i\binom{|E|}{i}(|G|-i)^{|X|}\quad .$$

Proof : (a) By Corollary 8.4, there is an injective map

$$\mu_X: S_T(X)\longrightarrow \prod_{W\in\mathcal{C}(E,X)} S_T(E)\,,\qquad \omega\mapsto \big(W\omega\big)_{W\in\mathcal{C}(E,X)}\,.$$

This is obviously right kA-linear. By Lemma 10.5, $S_T(X)$ is a free right kA-module and since the group algebra of a finite group is self-injective, $S_T(X)$ is an injective right kA-module. Therefore the kA-linear injection μ_X splits. Since tensoring with V preserves split injections, the map

$$\mu_X \otimes \mathrm{id}_V : S_T(X) \otimes_{kA} V \longrightarrow \prod_{W \in \mathcal{C}(E,X)} S_T(E) \otimes_{kA} V$$

is still injective.

Let J be a nonzero subfunctor of $S_{T,V}$ and let $\omega \in J(X)$ be nonzero for some finite set X. By injectivity of $\mu_X \otimes \mathrm{id}_V$, there exists $W \in \mathcal{C}(E,X)$ such that $u := W\omega \neq 0$. This is an element of $J(E) \subseteq S_{T,V}(E) = S_T(E) \otimes_{kA} V$. Writing $M = S_T(E)$ and $m_\sigma = \Delta_\sigma \widehat{i}$ for simplicity, recall from Theorem 7.4 that M has a k-basis $\{m_\sigma \mid \sigma \in \Sigma_E\}$ and that it is isomorphic to the fundamental $k\mathcal{R}_E$-module $M_{E,R^{op}}$, where $\mathcal{R}_E = \mathcal{C}(E,E)$. Moreover, the right action of $\gamma \in A$ permutes freely the basis of M and satisfies

$$m_\sigma \cdot \gamma = (\Delta_\sigma \widehat{i}) \cdot \gamma = \Delta_\sigma \Delta_\gamma \widehat{i} = \Delta_{\sigma\gamma} \widehat{i} = m_{\sigma\gamma} \, .$$

Letting $[\Sigma_E/A]$ denote a set of representatives of cosets σA, we obtain

$$M \otimes_{kA} V = \bigoplus_{\sigma \in [\Sigma_E/A]} m_\sigma \otimes V \, ,$$

and in particular $u = \sum_{\sigma \in [\Sigma_E/A]} m_\sigma \otimes v_\sigma$ for some $v_\sigma \in V$. Since $u \neq 0$, we have $v_\pi \neq 0$ for some $\pi \in [\Sigma_E/A]$.

The action of relations on the fundamental $k\mathcal{R}_E$-module is described in (7.2). In particular the action of ${}^{\pi}R^{op}$ is given by

$${}^{\pi}R^{op}\, m_\sigma = \begin{cases} m_{\tau\sigma} & \text{if } \exists\, \tau \in \Sigma_E \text{ such that } \Delta \subseteq \Delta_{\tau^{-1}}\, {}^{\pi}R^{op} \subseteq {}^{\sigma}R^{op} \, , \\ 0 & \text{otherwise.} \end{cases}$$

The first containment relation implies $\Delta_\tau \subseteq {}^{\pi}R^{op}$. This is impossible if $\tau \neq \mathrm{id}$ because a nontrivial permutation cannot be contained in a partial order relation. So $\tau = \mathrm{id}$ and we are left with ${}^{\pi}R^{op} \subseteq {}^{\sigma}R^{op}$, which implies ${}^{\pi}R^{op} = {}^{\sigma}R^{op}$ because the cardinalities are equal. Thus $\sigma^{-1}\pi$ stabilizes R^{op}, hence $\sigma^{-1}\pi \in A$ by (10.3). Since σ and π are representatives in $[\Sigma_E/A]$, we deduce that $\pi = \sigma$. In other words

$${}^{\pi}R^{op} m_\sigma = \begin{cases} m_\sigma & \text{if } \sigma = \pi \, , \\ 0 & \text{otherwise.} \end{cases}$$

Going back to $u = \sum_\sigma m_\sigma \otimes v_\sigma$, we get ${}^{\pi}R^{op} u = m_\pi \otimes v_\pi$. This is a nonzero element of $J(E)$ because $u \in J(E)$.

Take now $\gamma \in A$ and $\tau = \pi\gamma\pi^{-1} \in \Sigma_E$. Then

$$\Delta_\tau m_\pi \otimes v_\pi = m_{\tau\pi} \otimes v_\pi = m_{\pi\gamma} \otimes v_\pi = m_\pi \cdot \gamma \otimes v_\pi = m_\pi \otimes \gamma \cdot v_\pi \, .$$

Therefore $m_\pi \otimes \gamma \cdot v_\pi \in J(E)$ for every $\gamma \in A$. Since the kA-module V is simple, it is generated by its nonzero element v_π and we deduce that $m_\pi \otimes V \subseteq J(E)$. Now for any $\sigma \in [\Sigma_E/A]$, we obtain $m_\sigma \otimes V = \Delta_{\sigma\pi^{-1}} m_\pi \otimes V \subseteq J(E)$, hence $M \otimes_{kA} V \subseteq J(E)$. In other words $J(E) = S_T(E) \otimes_{kA} V = S_{T,V}(E)$.

In particular $\widehat{i} \otimes V \subseteq J(E)$ and therefore, for any correspondence $U \in \mathcal{C}(X,E)$, we get

$$U \widehat{i} \otimes V = U(\widehat{i} \otimes V) \subseteq U\, J(E) \subseteq J(X) \, .$$

It follows that $S_T(X) \otimes_{kA} V \subseteq J(X)$, so that $J(X) = S_{T,V}(X)$, hence $J = S_{T,V}$. This completes the proof of the simplicity of $S_{T,V}$.

(b) Let S be a simple correspondence functor and let E be a set of minimal cardinality such that $S(E) \neq \{0\}$. Then $S(E)$ is a $k\mathcal{R}_E$-module, where $\mathcal{R}_E = \mathcal{C}(E,E)$ is the set of all relations on E. Let $I = \sum_{|X|<|E|} k\mathcal{C}(E,X)\mathcal{C}(X,E)$

be the ideal generated by all relations which factorize through a set of cardinality $< |E|$. Since $S(X) = \{0\}$ whenever $|X| < |E|$, $S(E)$ is a $k\mathcal{R}_E/I$-module, where $k\mathcal{R}_E/I$ is the algebra of so-called essential relations on E. By Theorem 8.1 in [BT1], $S(E) \cong M \otimes_{kA} V$, where V is a simple left kA-module and M is a suitable $(k\mathcal{R}_E/I, kA)$-bimodule. More precisely, the left action of $k\mathcal{R}_E/I$ on M is described in Proposition 8.5 of [BT1], which asserts that M is isomorphic to the fundamental $k\mathcal{R}_E$-module of Section 7.

Take now any finite lattice T with $\mathrm{Mirr}(T) \cong (E, R)$ (e.g. the lattice of all down-sets in E, see Section 11). Then we know that $S_T(E) \cong M$ by Theorem 7.4, so there are isomorphisms of $k\mathcal{R}_E$-modules

$$S(E) \cong M \otimes_{kA} V \cong S_T(E) \otimes_{kA} V \cong S_{T,V}(E) .$$

But if two simple correspondence functors have isomorphic nonzero evaluations at some set E, they are isomorphic functors. This is a well-known general fact of the representation theory of small categories, which goes back to [Bo1] and is explicit in Proposition 2.7 of [BT2] or in Proposition 3.2 of [We]. Therefore we obtain $S \cong S_{T,V}$, as was to be shown.

(c) A simple kA-module V is generated by a single nonzero element $v_0 \in V$. It follows that any element of $S_T(X) \otimes_{kA} V$ can be written $\varphi \otimes v_0$ for some $\varphi \in S_T(X)$. Therefore we get a surjective morphism

$$S_T \longrightarrow S_{T,V} = S_T \otimes_{kA} V , \qquad \varphi \mapsto \varphi \otimes v_0 .$$

This shows that $S_{T,V}$ is isomorphic to a quotient of S_T.

(d) Since $S_T(X)$ is a free right $k\,\mathrm{Aut}(E, R)$-module (Lemma 10.5), we get

$$\dim_k S_{T,V}(X) = \frac{\dim_k(V)}{|\,\mathrm{Aut}(E, R)|} \dim_k S_T(X)$$

and the result follows from the formula in Theorem 6.3. □

Note that, by the formula in (d), we recover the main result for simple correspondence functors obtained in Theorem 7.10 of [BT4].

It should be noted that we may have two nonisomorphic lattices T and T' such that $S_{T,V} \cong S_{T',V}$, but T and T' must have isomorphic subposets $\mathrm{Mirr}(T) \cong (E, R) \cong \mathrm{Mirr}(T')$. More precisely, simple correspondence functors are parametrized by triples (E, R, V), independently of the choice of T with $\mathrm{Mirr}(T) \cong (E, R)$. We discuss this question in the next Section 11.

11. Choosing the lattice

As mentioned above, the choice of a lattice T is irrelevant for the parametrization of simple functors. The purpose of this section is to clarify this question. We prove that several constructions of the present paper do not depend on the choice of a finite lattice T, but only on the poset $\mathrm{Mirr}(T) \cong (E, R)$.

We fix a finite poset (E, R). Recall that a down-set in E is a subset F of E such that, for any $f \in F$ and any $e \in E$ with $e \leq f$, we have $e \in F$. Let D be the set of all down-sets in E. This is a lattice with respect to union and intersection of subsets (and it is actually a distributive lattice). For any $e \in E$, define

$$\underline{e} = \{f \in E \mid f \leq e\} , \qquad \underline{E} = \{\underline{e} \mid e \in E\} ,$$
$$e^* = \{f \in E \mid f \not\geq e\} , \qquad E^* = \{e^* \mid e \in E\} .$$

It is elementary to check that $\underline{E} = \mathrm{Jirr}(D)$, because any down-set X is a union $X = \bigcup_{e \in X} \underline{e}$ and each $\underline{e}$ is join-irreducible. Similarly, any up-set X is a union $X = \bigcup_{e \in X} E_{\geq e}$ and by passing to the complement we get $E - X = \bigcap_{e \in X} e^*$, because

$E - E_{\geq e} = \{f \in E \mid f \ngeq e\} = e^*$. Therefore $E^* = \mathrm{Mirr}(D)$. Moreover, for all $e, f \in E$,

$$e \leq f \iff \underline{e} \subseteq \underline{f} \iff e^* \subseteq f^* .$$

In other words, the maps $e \mapsto \underline{e}$ and $e \mapsto e^*$ are poset isomorphisms $E \cong \underline{E} \cong E^*$.

Throughout this paper, we used meet-irreducible elements and here again it is $E^* = \mathrm{Mirr}(D)$ which comes into play.

11.1. Lemma. *Let T be a finite lattice such that $\mathrm{Mirr}(T) = (E, R)$ and let D be the lattice of all down-sets in E. The map*

$$\alpha : T \longrightarrow D , \qquad \alpha(t) = \{f \in E \mid f \ngeq t\}$$

is a full join-morphism whose restriction to E is the isomorphism $E \cong E^*$. In particular α is injective.

Proof : If $t_1, t_2 \in T$, we have $f \geq (t_1 \vee t_2)$ if and only if $f \geq t_1$ and $f \geq t_2$. Therefore

$$\begin{aligned} \alpha(t_1 \vee t_2) &= \{f \in E \mid f \ngeq (t_1 \vee t_2)\} \\ &= \{f \in E \mid f \ngeq t_1\} \cup \{f \in E \mid f \ngeq t_2\} \\ &= \alpha(t_1) \cup \alpha(t_2) . \end{aligned}$$

Moreover, $\alpha(\widehat{0}_T) = \{f \in E \mid f \ngeq \widehat{0}_T\} = \emptyset = \widehat{0}_D$, so α is a join-morphism. To prove that α is full, suppose that $\alpha(t_1) \supseteq \alpha(t_2)$. The negation of the conditions yields $\{f \in E \mid f \geq t_1\} \subseteq \{f \in E \mid f \geq t_2\}$, hence

$$\bigwedge_{\substack{f \in E \\ f \geq t_1}} f \geq \bigwedge_{\substack{f \in E \\ f \geq t_2}} f .$$

By (2.3), any $t \in T$ is the meet of all meet-irreducible elements larger than t. Therefore we obtain $t_1 \geq t_2$, proving the α is full. In particular, α is injective.

The definition of α yields $\alpha(e) = \{f \in E \mid f \ngeq e\} = e^*$, for any $e \in E$, so the restriction of α to E is the isomorphism $E \cong E^*$. □

For any finite lattice L such that $\mathrm{Mirr}(L) = (E, R)$, let $s_L : L \to L$ be the map defined in (2.2). Also, write ρ_L for the map $\rho : L \to L$ defined in (4.3). Let G_L be the subset G of L defined in (4.4), namely

(11.2) $$G_L = \bigvee E \sqcup \widetilde{G}_L , \text{ where } \widetilde{G}_L = \{h \in L \mid s_L^\infty \rho_L^\infty(h) = h, \, h \notin \bigvee E\} .$$

We view G_L as a full subposet of L.

11.3. Lemma. *Let T be a finite lattice such that $\mathrm{Mirr}(T) = (E, R)$ and let $\alpha : T \to D$ be the join-morphism of Lemma 11.1.*

- *(a)* $\alpha \, s_T = s_D \, \alpha$.
- *(b)* $\alpha \, \rho_T = \rho_D \, \alpha$.
- *(c)* $\alpha(G_T) = G_D$. *In particular, the posets G_T and G_D are isomorphic, so that G_T does not depend on T, but only on (E, R) up to isomorphism.*

Proof : (a) Let $t \in T$. Recall that $s_T(t) = t$ if and only if $t \notin \mathrm{Mirr}(T)$, while, if $t \in \mathrm{Mirr}(T)$, $s_T(t)$ is the unique minimal element in $T_{>t}$ and the interval $]t, s_T(t)[$ is empty.

If $t \notin E$. Then $\alpha(t) \notin E^*$ by injectivity of α. Therefore $s_T(t) = t$ and $s_D(\alpha(t)) = \alpha(t)$, so that $\alpha \, s_T(t) = s_D \, \alpha(t)$.

If $e \in E$, then $E_{\geq e} = E_{>e} \sqcup \{e\} = E_{\geq s_T(e)} \sqcup \{e\}$. Therefore $E - E_{\geq s_T(e)} = (E - E_{\geq e}) \sqcup \{e\}$. In other words, $\alpha(s_T(e)) = \alpha(e) \sqcup \{e\} = e^* \sqcup \{e\}$. On the other hand $s_D(\alpha(e)) = s_D(e^*) = e^* \sqcup \{e\}$ because if $f^* \in E^*$ satisfies $f^* \supset e^*$, then $f > e$ and $e \in f^*$, hence $f^* \supseteq e^* \sqcup \{e\}$, showing that $e^* \sqcup \{e\}$ is the minimal element of $D_{\supset e^*}$. It follows that $\alpha(s_T(e)) = s_D(\alpha(e))$.

(b) Let $f \in E$. If $f \leq t$, then $f^* \subseteq \alpha(t)$. Conversely, if $f^* \subseteq \alpha(t)$, then $(E-E_{\geq f}) \subseteq (E-E_{\geq t})$, hence $E_{\geq f} \supseteq E_{\geq t}$. Since t is the meet of all meet-irreducible elements larger than t, by (2.3), we get

$$f = \bigwedge_{e \geq f} e \leq \bigwedge_{e \geq t} e = t\,,$$

so that $f \leq t$. This shows that $f < t$ if and only if $f^* \subset \alpha(t)$. Therefore

$$\alpha(\rho_T(t)) = \alpha(\bigvee_{f<t} f) = \bigcup_{f<t} \alpha(f) = \bigcup_{f^* \subset \alpha(t)} f^* = \rho_D(\alpha(t))\,,$$

as was to be shown.

(c) Since $\alpha(E) = E^*$, we get $\alpha(\bigvee E) = \bigvee E^*$. Now applying (a) and (b) to the characterization of $\widetilde{G}_T$ recalled in (11.2) above, we see immediately that $\alpha(\widetilde{G}_T) = \widetilde{G}_D$ and therefore $\alpha(G_T) = G_D$. Since G_D is a full subposet of D and α is full (by Lemma 11.1), we see that $\alpha_{|G_T} : G_T \to G_D$ is an isomorphism of posets. Now D only depends on (E, R), by its very definition, so it is clear that $G_T \cong G_D$ only depends on (E, R). ∎

The fact that the poset G_T does not depend on T, but only on (E, R) up to isomorphism, was left without proof in Corollary 6.7 of [BT4]. This was settled in Theorem 4.2 of [Bo2], as a by-product of a far reaching generalization of the construction of G. The proof above shows that the result on its own is in fact rather elementary.

Since the map α of Lemma 11.1 is a join-morphism, it induces a morphism of correspondence functors $\alpha : F_T \to F_D$ mapping $\varphi \in T^X$ to $\alpha\varphi \in D^X$.

11.4. Theorem. *Let T be a finite lattice such that* $\mathrm{Mirr}(T) = (E, R)$ *and let $\alpha : T \to D$ be the join-morphism of Lemma 11.1.*

(a) $\alpha : F_T \to F_D$ restricts to an isomorphism of functors $S_T \to S_D$.
(b) For any finite set X, the isomorphism $F_T(X) \to F_D(X)$ is also an isomorphism of right $k\,\mathrm{Aut}(E, R)$-modules.
(c) α induces an isomorphism of simple correspondence functors $S_{T,V} \to S_{D,V}$, for any simple left $k\,\mathrm{Aut}(E, R)$-module V. In particular, this simple functor does not depend on T, but only on (E, R) up to isomorphism.

Proof : (a) By Theorem 6.3, $S_T(X)$ has a k-basis $\widehat{\mathcal{B}}_X = \{\widehat{\varphi} \mid \varphi \in \mathcal{B}_X\}$, where $\mathcal{B}_X = \{\varphi \in T^X \mid E \subseteq \varphi(X) \subseteq G_T\}$. Since α maps $\varphi \in \mathcal{B}_X$ to the composition $\alpha\varphi$ and since $\alpha(E) = E^*$ (Lemma 11.1) and $\alpha(G_T) = G_D$ (Lemma 11.3), we see that α maps $\mathcal{B}_X$ to

$$\mathcal{B}^*_X := \{\alpha\varphi \in D^X \mid E^* \subseteq \alpha\varphi(X) \subseteq G_D\} = \{\psi \in D^X \mid E^* \subseteq \psi(X) \subseteq G_D\}\,.$$

Now we claim that $\widehat{\alpha\varphi} = \alpha\widehat{\varphi}$. Postponing the proof of the claim, it follows that

$$\begin{aligned} \alpha(\widehat{\mathcal{B}}_X) &= \{\alpha\widehat{\varphi} \mid \varphi \in \mathcal{B}_X\} = \{\widehat{\alpha\varphi} \mid \varphi \in \mathcal{B}_X\} = \{\widehat{\alpha\varphi} \mid \alpha\varphi \in \mathcal{B}^*_X\} \\ &= \{\widehat{\psi} \mid \psi \in \mathcal{B}^*_X\} = \widehat{\mathcal{B}}^*_X\,. \end{aligned}$$

Theorem 6.3 asserts that $\widehat{\mathcal{B}}^*_X$ is a basis of $S_D(X)$, so we see that α maps bijectively a basis of $S_T(X)$ to a basis of $S_D(X)$. Therefore the restriction of α is an isomorphism $S_T \to S_D$.

To prove the claim, we first show that $\alpha\varphi_A = (\alpha\varphi)_{A^*}$ where $A^* = \alpha(A)$ for any $A \subseteq E$. This is because

$$\alpha\varphi_A(x) = \begin{cases} \alpha\varphi(x) & \text{if } \varphi(x) \notin A\,, \\ \alpha s_T\varphi(x) & \text{if } \varphi(x) \in A\,, \end{cases}$$

while

$$(\alpha\varphi)_{A^*}(x) = \begin{cases} \alpha\varphi(x) & \text{if } \alpha\varphi(x) \notin A^*\,, \\ s_D\alpha\varphi(x) & \text{if } \alpha\varphi(x) \in A^*\,, \end{cases}$$

so that $\alpha\varphi_A = (\alpha\varphi)_{A^*}$ since $\alpha s_T = s_D\alpha$ by Lemma 11.3. It follows that

$$\alpha\widehat{\varphi} = \sum_{A\subseteq E} (-1)^{|A|}\alpha\varphi_A = \sum_{A^*\subseteq E^*} (-1)^{|A^*|}(\alpha\varphi)_{A^*} = \widehat{\alpha\varphi}\,,$$

proving the claim.

(b) The isomorphism $\alpha : S_T(E) \to S_D(E)$ maps $\widehat{i}$ to $\alpha\widehat{i}$, which is equal to the generator $\widehat{\alpha\, i}$ of $S_D(E)$, by the claim above. It follows that the isomorphism $S_T(X) \to S_D(X)$ maps $U\,\widehat{i}$ to $U\widehat{\alpha\, i}$. By the definition of the right action of $\gamma \in \mathrm{Aut}(E,R)$ (see Lemma 10.4), we obtain

$$\alpha((U\,\widehat{i})\cdot\gamma) = \alpha(U\Delta_\gamma\,\widehat{i}) = U\Delta_\gamma\,\alpha\,\widehat{i} = U\Delta_\gamma\,\widehat{\alpha\, i} = (U\,\widehat{\alpha\, i})\cdot\gamma = \alpha(U\,\widehat{i})\cdot\gamma\,,$$

as required.

(c) Since $S_{T,V} = S_T \otimes_{kA} V$ and $S_{D,V} = S_D \otimes_{kA} V$, where $A = \mathrm{Aut}(E,R)$, the isomorphism $S_{T,V} \to S_{D,V}$ follows from (a) and (b). ∎

11.5. Remark. In the proof of (b), we have used $S_D(E)$, whereas, since $\mathrm{Mirr}(D) = E^*$, the functor S_D is naturally generated by $S_D(E^*)$ instead of $S_D(E)$. But one can pass from $S_D(E)$ to $S_D(E^*)$ by left composition with the correspondence $J = \{(e^*,e) \mid e \in E\}$, which realizes the isomorphism $\alpha_{|E} : E \to E^*$. Explicitly, the computation gives, for any $\varphi \in D^E$,

$$(J\varphi)(e^*) = \varphi(e) = \varphi\,\alpha_{|E}^{-1}(e^*)\,,$$

and we see that we have pre-composed by the inverse of this isomorphism. As a result, S_D is naturally generated by $\widehat{i}^* := \alpha\,\widehat{i}\,\alpha_{|E}^{-1} \in S_D(E^*)$.

Theorem 10.6 and Theorem 11.4 imply that we recover the parametrization of simple functors obtained in [BT2], but with a rather different approach.

11.6. Corollary. *The simple correspondence functors are parametrized by isomorphism classes of triples (E,R,V), where (E,R) is a finite poset and V is a simple $k\,\mathrm{Aut}(E,R)$-module.*

12. Functors associated to a join-morphism

Associated to any join-morphism of finite lattices, there is a duality studied in [BT5] giving rise to more correspondence functors. The purpose of this section is to establish a link between this construction and the approach of the present paper.

The starting point in [BT5] is a join-morphism $\alpha : T \to T'^{op}$, where T and T' are finite lattices, together with the subset

$$\Phi_\alpha = \{f \in \mathrm{Jirr}(T) \mid \alpha(f) \in \mathrm{Jirr}(T'), \alpha^{op}\alpha(f) = f\}\,.$$

Note that the definition of α uses T'^{op} whereas Φ_α uses T'. In view of the approach of the present paper, it is convenient to set $L = T'^{op}$, so that α is simply a join-morphism from T to L. As a result, one gets $\mathrm{Jirr}(T') = \mathrm{Jirr}(L^{op}) = \mathrm{Mirr}(L)$, and therefore

$$\begin{aligned}\Phi_\alpha &= \{f \in \mathrm{Jirr}(T) \mid \alpha(f) \in \mathrm{Mirr}(L), \alpha^{op}\alpha(f) = f\}\,,\\ \Phi'_\alpha &= \{f' \in \mathrm{Mirr}(L) \mid \alpha^{op}(f') \in \mathrm{Jirr}(T), \alpha\alpha^{op}(f') = f'\}\,.\end{aligned}$$

This is the notation used in [BT5]. For simplicity, we shall define $E := \Phi_\alpha$ and $E' := \Phi'_\alpha$.

12.1. Lemma. *Let $\alpha : T \to L$ be a join-morphism of finite lattices. Let E, viewed as a full subposet of T, and E', viewed as a full subposet of L, be defined as above. The restriction of α to E yields an isomorphism of posets $E \to E'$. Its inverse is induced by the restriction of α^{op} to E'.*

Proof : This is Lemma 3.6 in [BT5], observing that $\alpha : T \to L$ is order-preserving, whereas in [BT5], α was viewed as an order-reversing map $\alpha : T \to T'$. ∎

We also extend the notation to elements of E and E' and to subsets of E and E', as follows:

$$\text{if } e \in E, \text{ then } e' = \alpha(e) \in E', \text{ and if } e' \in E', \text{ then } e = \alpha^{op}(e') \in E\,.$$

Similarly,

$$\text{if } B \subseteq E, \text{ then } B' = \alpha(B) \subseteq E', \text{ and if } B' \subseteq E', \text{ then } B = \alpha^{op}(B') \subseteq E\,.$$

If $\alpha(f) \in \mathrm{Jirr}(T') = \mathrm{Mirr}(L)$, the unique maximal element of the interval $[0_{T'}, \alpha(f)[$ in T' is $r_{T'}(\alpha(f))$, which is of course equal to $s_L(\alpha(f))$, the unique minimal element of the interval $]\alpha(f), 1_L]$ in L. Working with $\mathrm{Mirr}(L)$ and the operator s_L is precisely the point of view used in the present paper.

Associated with a given join-morphism $\alpha : T \to L$, there is a correspondence functor $\mathbb{S}_\alpha$ obtained by means of a certain duality which is a main tool in [BT5]. But it is shown in Corollary 3.15 of [BT5] that $\mathbb{S}_\alpha$ can also be described in a different way (up to isomorphism). Using this alternative description, we define

$$\mathbb{S}_\alpha = \breve{\alpha}(F_T)\,,$$

the image of the morphism $\breve{\alpha} : F_T \to F_L$ induced by the following linear combination of join-morphisms from T to L :

$$\breve{\alpha} = \sum_{A\subseteq E} (-1)^{|A|}\alpha_A\,,$$

where α_A is defined by

$$\alpha_A(t) = \begin{cases} s_L\alpha(t) & \text{if } t \in A\,,\\ \alpha(t) & \text{if } t \in T{-}A\,.\end{cases}$$

This sums resembles the definition of $\widehat{\varphi}^E$ in (3.8) and the purpose of this section is to explore this resemblance. More precisely, for any $\varphi \in T^X$, we have $\alpha\varphi \in L^X$ and $\widehat{\alpha\varphi}^{E'}$ resembles $\breve{\alpha}\varphi$, but we shall see that it is not always equal to $\breve{\alpha}\varphi$. As before, for any map $\psi : X \to L$, we write for simplicity $\widehat{\psi}^{E'} = \widehat{\psi}$.

First define, for all $t \in T$,

$$\overline{t} = \begin{cases} \alpha^{op}\alpha(t) & \text{if } \alpha(t) \in E'\,,\\ t & \text{if } \alpha(t) \in L{-}E'\,.\end{cases}$$

For any $\varphi \in T^X$, one defines also $\overline{\varphi} \in T^X$ by $\overline{\varphi}(x) = \overline{\varphi(x)}$ for all $x \in X$. Notice that $\alpha\overline{\varphi} = \alpha\varphi$ because $\alpha\alpha^{op}\alpha = \alpha$.

12.2. Lemma. *Let $\varphi \in T^X$.*

(a) If $A \subseteq E$ and $A' = \alpha(A) \subseteq E'$, we have an equality $(\alpha\varphi)_{A'} = \alpha_A\overline{\varphi}$.
(b) $\widehat{\alpha\varphi} = \breve{\alpha}\overline{\varphi}$.
(c) If $\varphi(x) \in T{-}A$ but $\alpha\varphi(x) \in A'$, one has $(\alpha\varphi)_{A'}(x) \neq \alpha_A\varphi(x)$.

Proof : (a) Let $x \in X$. By Definition 3.7, one gets

$$(\alpha\varphi)_{A'}(x) = \begin{cases} s_L\alpha\varphi(x) & \text{if } \alpha\varphi(x) \in A' \,, \\ \alpha\varphi(x) & \text{if } \alpha\varphi(x) \in L{-}A' \,. \end{cases}$$

On the other hand, the definition of α_A given above and the fact that $\alpha\overline{\varphi} = \alpha\varphi$ imply

$$\alpha_A\overline{\varphi}(x) = \begin{cases} s_L\alpha\overline{\varphi}(x) = s_L\alpha\varphi(x) & \text{if } \overline{\varphi}(x) \in A \,, \\ \alpha\overline{\varphi}(x) = \alpha\varphi(x) & \text{if } \overline{\varphi}(x) \in T{-}A \,. \end{cases}$$

The condition $\overline{\varphi}(x) \in A$ yields either $\alpha^{op}\alpha\varphi(x) \in A$ or $\varphi(x) \in A$, according to the definition of $\overline{\varphi}$. Applying α and using $\alpha\alpha^{op}\alpha = \alpha$, we see that $\alpha\varphi(x) \in A'$ in both cases. Conversely, if $\alpha\varphi(x) \in A'$, then $\alpha\varphi(x) \in E'$ and the first case of the definition of $\overline{\varphi}(x)$ yields $\overline{\varphi}(x) = \alpha^{op}\alpha\varphi(x) \in A$. We see that the conditions correspond to each other, proving the equality $(\alpha\varphi)_{A'} = \alpha_A\overline{\varphi}$.

(b) By (a), one obtains

$$\widehat{\alpha\varphi} = \sum_{A' \subseteq E'} (-1)^{|A'|}(\alpha\varphi)_{A'} = \sum_{A \subseteq E} (-1)^{|A|}\alpha_A\overline{\varphi} = \breve{\alpha}\overline{\varphi} \,.$$

(c) When $\varphi(x) \in T{-}A$, one has $\alpha_A\varphi(x) = \alpha\varphi(x)$, while when $\alpha\varphi(x) \in A'$, one gets $(\alpha\varphi)_{A'}(x) = s_L\alpha\varphi(x) \neq \alpha\varphi(x)$. □

Note that case (c) in Lemma 12.2 arises for instance when $t := \varphi(x) \notin E$ but $\alpha(t) \in E'$. In the notation of Lemma 4.7 and Proposition 4.9 in [BT5], one gets $\overline{t} = \beta\alpha(t)$ and $t < \beta\alpha(t) \in E$, where $\beta = \alpha^{op}$. The distinction between t and $\overline{t}$ is the main ingredient in Lemma 12.2.

Actually, if we had $\overline{t} = t$ for all $t \in T$, then $\breve{\alpha}\varphi$ would be equal to $\widehat{\alpha\varphi}$, which belongs to $Q_L^{E'}$ by Lemma 3.9. In that case, it would follow that $\mathbb{S}_\alpha = \breve{\alpha}(F_T)$ is a subfunctor of F_L contained in $Q_L^{E'}$. We now show that this containment in fact always holds.

12.3. Theorem. *The subfunctor $\mathbb{S}_\alpha = \breve{\alpha}(F_T)$ of F_L has the following properties :*

(a) $\mathbb{S}_\alpha$ contains $S_L^{E'}$.
(b) $\mathbb{S}_\alpha$ is contained in $Q_L^{E'}$. Explicitly, $\widehat{\breve{\alpha}\varphi} = \breve{\alpha}\varphi$ for any function $\varphi \in T^X$.

Proof : (a) Let $j : E' \to L$ be the inclusion map. By definition, $S_L^{E'}$ is generated by $\widehat{j} \in F_L(E')$. Since α induces a bijection $\delta : E \to E'$ by Lemma 12.1, $S_L^{E'}$ is also generated by $\widehat{j}\,\delta \in F_L(E)$. But we have clearly $\widehat{j}\,\delta = \widehat{j\delta}$ (see Lemma 6.1). If $i : E \to T$ denotes the inclusion map, we have $j\delta = \alpha i$, because the evaluation at $e \in E$ gives $\alpha(e)$ in both cases (since $\delta(e) = \alpha(e)$ by definition). By Lemma 12.2, we obtain

$$\widehat{j\delta} = \widehat{\alpha i} = \breve{\alpha}\overline{i} \,.$$

But $\overline{i} = i$ because $\overline{i}(e) = \alpha^{op}\alpha i(e) = \delta^{-1}\delta(e) = e = i(e)$ for all $e \in E$. It follows that

$$\widehat{j}\,\delta = \widehat{j\delta} = \breve{\alpha}\overline{i} = \breve{\alpha}i$$

and therefore the generator $\widehat{j}\,\delta$ of $S_L^{E'}$ belongs to $\breve{\alpha}(F_T)(E) = \mathbb{S}_\alpha(E)$.

(b) We have to prove that $\breve{\alpha}\varphi$ belongs to $Q_L^{E'}(X)$ for any $\varphi \in T^X$. By Proposition 3.17, we need to show that $\widehat{\breve{\alpha}\varphi} = \breve{\alpha}\varphi$. But $\breve{\alpha} = \sum_{B\subseteq E}(-1)^{|B|}\alpha_B$, so that

$$\begin{aligned}
\widehat{\breve{\alpha}\varphi} &= \sum_{B\subseteq E}(-1)^{|B|}\widehat{\alpha_B\,\varphi} = \sum_{B\subseteq E}(-1)^{|B|}\sum_{A'\subseteq E'}(-1)^{|A'|}(\alpha_B\,\varphi)_{A'} \\
&= \sum_{B\subseteq E}(-1)^{|B|}\alpha_B\,\varphi + \sum_{B\subseteq E,\,A'\neq\emptyset}(-1)^{|B|+|A'|}(\alpha_B\,\varphi)_{A'} \\
&= \breve{\alpha}\varphi + \sum_{B\subseteq E,\,A'\neq\emptyset}(-1)^{|B|+|A'|}(\alpha_B\,\varphi)_{A'}\,.
\end{aligned}$$

We are going to prove that the latter sum is zero. We fix a nonempty subset A' of E' and we intend to show that the sum $\sum_{B\subseteq E}(-1)^{|B|}(\alpha_B\,\varphi)_{A'}$ is zero. We define

$$M' = \{e' \in A' \mid s_L(e') \notin A'\}$$

and we note that all maximal elements of A' belong to M', so that M' is nonempty since A' is nonempty. Using again the bijection of Lemma 12.1, recall that $A = \alpha^{op}A'$ and $M = \alpha^{op}M'$.

Any subset $B \subseteq E$ is a disjoint union $B = C \cup D$ with $C \subseteq E{-}M$ and $D \subseteq M$. Therefore

$$\sum_{B\subseteq E}(-1)^{|B|}(\alpha_B\,\varphi)_{A'} = \sum_{C\subseteq E-M}(-1)^{|C|}\big(\sum_{D\subseteq M}(-1)^{|D|}(\alpha_{C\cup D}\,\varphi)_{A'}\big)$$

and it suffices to prove that, for any fixed subset $C \subseteq E{-}M$, the inner sum $\sum_{D\subseteq M}(-1)^{|D|}(\alpha_{C\cup D}\,\varphi)_{A'}$ is zero. By Lemma 12.4 below,

$$\sum_{D\subseteq M}(-1)^{|D|}(\alpha_{C\cup D}\,\varphi)_{A'} = \sum_{D\subseteq M}(-1)^{|D|}(\alpha_C\,\varphi)_{A'}\,.$$

But $\sum_{D\subseteq M}(-1)^{|D|} = 0$, because M is nonempty. Therefore

$$\sum_{D\subseteq M}(-1)^{|D|}(\alpha_{C\cup D}\,\varphi)_{A'} = 0$$

as required. This will complete the proof of Theorem 12.3, provided we prove Lemma 12.4 below. ∎

12.4. Lemma. *Let M be as above and let C be a subset of $E{-}M$. For any subset D of M, the functions $(\alpha_{C\cup D}\,\varphi)_{A'}$ and $(\alpha_C\,\varphi)_{A'}$ are equal.*

Proof : Let $x \in X$. We first compute $(\alpha_C\,\varphi)_{A'}(x)$ by discussing the values of $\varphi(x)$.

If $\varphi(x) \in C$, we have $\alpha_C\,\varphi(x) = s_L\alpha\varphi(x)$. But

$$\varphi(x) \in C \Rightarrow \varphi(x) \notin M \Rightarrow \alpha\varphi(x) \notin M' \Rightarrow s_L\alpha\varphi(x) \in A'$$

by the definition of M'. Thus $\alpha_C\,\varphi(x) \in A'$ and we obtain :

(12.5) If $\varphi(x) \in C$, $\alpha_C\,\varphi(x) \in A'$ and $(\alpha_C\,\varphi)_{A'}(x) = s_L\alpha_C\,\varphi(x) = s_Ls_L\alpha\varphi(x)$.

If now $\varphi(x) \notin C$, we get $\alpha_C\,\varphi(x) = \alpha\varphi(x)$, hence :

(12.6) $$\text{If } \varphi(x) \notin C\,,\ (\alpha_C\,\varphi)_{A'}(x) = \begin{cases} \alpha\varphi(x) & \text{if } \alpha\varphi(x) \notin A'\,, \\ s_L\alpha\varphi(x) & \text{if } \alpha\varphi(x) \in A'\,. \end{cases}$$

Now we compute $(\alpha_{C\cup D}\,\varphi)_{A'}(x)$ by discussing again the values of $\varphi(x)$.

If $\varphi(x) \in C$, we have $\alpha_{C\cup D}\,\varphi(x) = s_L\alpha\varphi(x) = \alpha_C\,\varphi(x)$ and this belongs to A', as noticed in (12.5). It follows that :

(12.7) $$\text{If } \varphi(x) \in C\,,\ (\alpha_{C\cup D}\,\varphi)_{A'}(x) = s_L\alpha_{C\cup D}\,\varphi(x) = s_Ls_L\alpha\varphi(x)\,.$$

If $\varphi(x) \in D$, we have $\alpha_{C\cup D}\,\varphi(x) = s_L\alpha\varphi(x)$. But

$$\varphi(x) \in D \Rightarrow \varphi(x) \in M \Rightarrow \alpha\varphi(x) \in M' \Rightarrow \alpha\varphi(x) \in A' \text{ and } s_L\alpha\varphi(x) \notin A'$$

by the definition of M'. Thus we obtain :

(12.8) If $\varphi(x) \in D$, $\alpha\varphi(x) \in A'$ and $(\alpha_{C\cup D}\,\varphi)_{A'}(x) = (s_L\alpha\varphi)_{A'}(x) = s_L\alpha\varphi(x)$.

Finally, if $\varphi(x) \notin C \cup D$, we get $\alpha_{C\cup D}\,\varphi(x) = \alpha\varphi(x)$, hence :

(12.9) If $\varphi(x) \notin C \cup D$, $(\alpha_{C\cup D}\,\varphi)_{A'}(x) = \begin{cases} \alpha\varphi(x) & \text{if } \alpha\varphi(x) \notin A', \\ s_L\alpha\varphi(x) & \text{if } \alpha\varphi(x) \in A'. \end{cases}$

The comparison between (12.5) and (12.7) in case $\varphi(x) \in C$, respectively between (12.6), (12.8) and (12.9) in case $\varphi(x) \notin C$, shows that $(\alpha_C\,\varphi)_{A'}(x) = (\alpha_{C\cup D}\,\varphi)_{A'}(x)$. This completes the proof of Lemma 12.4 and so Theorem 12.3 is now fully established. $\square$

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Serge Bouc, CNRS-LAMFA, Université de Picardie - Jules Verne,
33, rue St Leu, F-80039 Amiens Cedex 1, France.
serge.bouc@u-picardie.fr

Jacques Thévenaz, Institut de mathématiques, EPFL,
Station 8, CH-1015 Lausanne, Switzerland.
jacques.thevenaz@epfl.ch