Fused Mackey functors

Serge Bouc

CNRS-LAMFA
Université de Picardie

Antalya Algebra Days - Şirince - 24/05/2013

Mackey functors (Green)

Mackey functors

A Mackey functor M for a finite group G

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t_K^K : M(H) \to M(K)$.

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t_K^K : M(H) \to M(K)$.
- for $g \in G$ and $H \leq G$, a conjugation homomorphism $c_{g,H} : M(H) \to M({}^g H)$.

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t_K^K : M(H) \to M(K)$.
- for $g \in G$ and $H \leq G$, a conjugation homomorphism $c_{g,H} : M(H) \to M({}^gH)$.

These data are subject to transitivity conditions

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t_K^K : M(H) \to M(K)$.
- for $g \in G$ and $H \leq G$, a conjugation homomorphism $c_{g,H} : M(H) \to M({}^gH)$.

These data are subject to transitivity conditions (for example $N_G(H)$ acts on $M(H)$, for any $H \leq G$)

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t_K^K : M(H) \to M(K)$.
- for $g \in G$ and $H \leq G$, a conjugation homomorphism $c_{g,H} : M(H) \to M({}^g H)$.

These data are subject to transitivity conditions, triviality conditions

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t_K^K : M(H) \to M(K)$.
- for $g \in G$ and $H \leq G$, a conjugation homomorphism $c_{g,H} : M(H) \to M({}^g H)$.

These data are subject to transitivity conditions, triviality conditions (for example H acts trivially on $M(H)$, for any $H \leq G$)

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t_K^K : M(H) \to M(K)$.
- for $g \in G$ and $H \leq G$, a conjugation homomorphism $c_{g,H} : M(H) \to M({}^gH)$.

These data are subject to transitivity conditions, triviality conditions, and compatibility conditions

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t_K^K : M(H) \to M(K)$.
- for $g \in G$ and $H \leq G$, a conjugation homomorphism $c_{g,H} : M(H) \to M({}^g H)$.

These data are subject to transitivity conditions, triviality conditions, and compatibility conditions, e.g. the Mackey formula

$$\begin{array}{ccc} & G & \\ & L & \\ \forall\, K & & H \end{array}$$

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r^K_H : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t^K_K : M(H) \to M(K)$.
- for $g \in G$ and $H \leq G$, a conjugation homomorphism $c_{g,H} : M(H) \to M({}^gH)$.

These data are subject to transitivity conditions, triviality conditions, and compatibility conditions, e.g. the Mackey formula

$$\forall \begin{array}{ccccc} & & G & & \\ & & | & & \\ & & L & & \\ & / & & \backslash & \\ K & & & & H \end{array}, \quad r^L_K \circ t^L_H =$$

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t_K^K : M(H) \to M(K)$.
- for $g \in G$ and $H \leq G$, a conjugation homomorphism $c_{g,H} : M(H) \to M({}^g H)$.

These data are subject to transitivity conditions, triviality conditions, and compatibility conditions, e.g. the Mackey formula

$$\begin{array}{ccc} & G & \\ & | & \\ & L & \\ \forall\ K & & H \end{array}, \quad r_K^L \circ t_H^L = \sum_{g \in [K \backslash L / H]}$$

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t_K^K : M(H) \to M(K)$.
- for $g \in G$ and $H \leq G$, a conjugation homomorphism $c_{g,H} : M(H) \to M({}^g H)$.

These data are subject to transitivity conditions, triviality conditions, and compatibility conditions, e.g. the Mackey formula

$$\forall \begin{tikzcd}[column sep=small] & G \arrow[d, dash] & \\ & L \arrow[dl, dash] \arrow[dr, dash] & \\ K & & H \end{tikzcd}, \quad r_K^L \circ t_H^L = \sum_{g \in [K \backslash L / H]} \qquad r_{K^g \cap H}^H$$

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t_K^K : M(H) \to M(K)$.
- for $g \in G$ and $H \leq G$, a conjugation homomorphism $c_{g,H} : M(H) \to M({}^g H)$.

These data are subject to transitivity conditions, triviality conditions, and compatibility conditions, e.g. the Mackey formula

$$\forall\ \begin{array}{ccc} & \color{gray}{G} & \\ & | & \\ & L & \\ / & & \backslash \\ K & & H \end{array}, \quad r_K^L \circ t_H^L = \sum_{g \in [K\backslash L/H]} c_{g,K^g \cap H} \circ r_{K^g \cap H}^H$$

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t_K^K : M(H) \to M(K)$.
- for $g \in G$ and $H \leq G$, a conjugation homomorphism $c_{g,H} : M(H) \to M({}^g H)$.

These data are subject to transitivity conditions, triviality conditions, and compatibility conditions, e.g. the Mackey formula

$$\forall \begin{array}{ccc} & G & \\ & \vdots & \\ & L & \\ / & & \backslash \\ K & & H \end{array}, \quad r_K^L \circ t_H^L = \sum_{g \in [K \backslash L / H]} t^K_{K \cap {}^g H} \circ c_{g, K^g \cap H} \circ r^H_{K^g \cap H} \quad .$$

Mackey functors

A Mackey functor M for a finite group G consists of the following data:

- for $H \leq G$, an abelian group $M(H)$.
- for $H \leq K \leq G$, a restriction homomorphism $r_H^K : M(K) \to M(H)$ and a transfer (or induction) homomorphism $t_K^K : M(H) \to M(K)$.
- for $g \in G$ and $H \leq G$, a conjugation homomorphism $c_{g,H} : M(H) \to M({}^g H)$.

These data are subject to transitivity conditions, triviality conditions, and compatibility conditions

Mackey functors for G form an abelian category $Mack(G)$.

Biset functors

Biset functors

A biset functor F

Biset functors

A biset functor F consists of the following data:

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K,H) of finite groups and any finite (K,H)-biset U, a group homomorphism $F(U): F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K,H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K,H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K,H)-biset

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H, then $F(Id_H) = Id_{F(H)}$.

Biset functors

A biset functor F consists of the following data:

- for each finite group H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H, then $F(Id_H) = Id_{F(H)}$.

Biset functors

A biset functor F consists of the following data:

- for some finite groups H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H, then $F(Id_H) = Id_{F(H)}$.

Biset functors

A biset functor F consists of the following data:

- for some finite groups H, an abelian group $F(H)$.
- for any pair (K, H) of finite groups and any finite (K, H)-biset U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H, then $F(Id_H) = Id_{F(H)}$.

Biset functors

A biset functor F consists of the following data:

- for some finite groups H, an abelian group $F(H)$.
- for any pair (K, H) of those finite groups

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H, then $F(Id_H) = Id_{F(H)}$.

Biset functors

A biset functor F consists of the following data:

- for some finite groups H, an abelian group $F(H)$.
- for any pair (K, H) of those finite groups and some finite (K, H)-bisets U

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H, then $F(Id_H) = Id_{F(H)}$.

Biset functors

A biset functor F consists of the following data:

- for some finite groups H, an abelian group $F(H)$.
- for any pair (K, H) of those finite groups and some finite (K, H)-bisets U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H, then $F(Id_H) = Id_{F(H)}$.

Biset functors

A biset functor F consists of the following data:

- for some finite groups H, an abelian group $F(H)$.
- for any pair (K, H) of those finite groups and some finite (K, H)-bisets U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H, then $F(Id_H) = Id_{F(H)}$.

Biset functors

A biset functor F consists of the following data:

- for some finite groups H, an abelian group $F(H)$.
- for any pair (K, H) of those finite groups and some finite (K, H)-bisets U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H, then $F(Id_H) = Id_{F(H)}$.

Example of "some":

Biset functors

A biset functor F consists of the following data:

- for some finite groups H, an abelian group $F(H)$.
- for any pair (K, H) of those finite groups and some finite (K, H)-bisets U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H, then $F(Id_H) = Id_{F(H)}$.

Example of "some": p-groups, for a prime number p.

Biset functors

A biset functor F consists of the following data:

- for some finite groups H, an abelian group $F(H)$.
- for any pair (K, H) of those finite groups and some finite (K, H)-bisets U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H, then $F(Id_H) = Id_{F(H)}$.

Example of "some": p-groups, for a prime number p.
Example of "some":

Biset functors

A biset functor F consists of the following data:

- for some finite groups H, an abelian group $F(H)$.
- for any pair (K, H) of those finite groups and some finite (K, H)-bisets U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H, then $F(Id_H) = Id_{F(H)}$.

Example of "some": p-groups, for a prime number p.
Example of "some": left-free bisets, or bi-free bisets.

Biset functors

A biset functor F consists of the following data:

- for some finite groups H, an abelian group $F(H)$.
- for any pair (K, H) of those finite groups and some finite (K, H)-bisets U, a group homomorphism $F(U) : F(H) \to F(K)$.

These data are subject to the following conditions:

1. if $U \cong U'$ as (K, H)-bisets, then $F(U) = F(U')$.
2. if $U = U_1 \sqcup U_2$ as (K, H)-bisets, then $F(U) = F(U_1) + F(U_2)$.
3. if V is a (K, H)-biset and U is an (H, G)-biset, then $F(V) \circ F(U) = F(V \circ U)$, where $V \circ U = V \times_H U$.
4. if Id_H is the (H, H)-biset H, then $F(Id_H) = Id_{F(H)}$.

Example of "some": p-groups, for a prime number p.
Example of "some": left-free bisets, or bi-free bisets.
Biset functors defined over "some" groups with "some" bisets as morphisms form an abelian category $\mathcal{F}_{some,some}$.

Conjugation bisets

Conjugation bisets

Fix a finite group G.

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow$$

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Let $Conj_{K,H}^G$ denote the full subcategory of (K, H)-bisets consisting of conjugation (K, H)-bisets

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Let $Conj_{K,H}^G$ denote the full subcategory of (K, H)-bisets consisting of conjugation (K, H)-bisets, and $\mathcal{B}_{K,H}^G$ its Grothendieck (Burnside) group.

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Let $Conj^G_{K,H}$ denote the full subcategory of (K, H)-bisets consisting of conjugation (K, H)-bisets, and $\mathcal{B}^G_{K,H}$ its Grothendieck (Burnside) group.

Examples:

- Let U be a sub-(K, H)-biset of ${}_K G_H$.

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^{g}h.$$

Let $Conj^{G}_{K,H}$ denote the full subcategory of (K, H)-bisets consisting of conjugation (K, H)-bisets, and $\mathcal{B}^{G}_{K,H}$ its Grothendieck (Burnside) group.

Examples:

- Let U be a sub-(K, H)-biset of ${}_{K}G_{H}$. Then $ku = uh \Leftrightarrow k = {}^{u}h$, for $u \in U$

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \;\; ku = uh \Rightarrow k = {}^g h.$$

Let $Conj^G_{K,H}$ denote the full subcategory of (K, H)-bisets consisting of conjugation (K, H)-bisets, and $\mathcal{B}^G_{K,H}$ its Grothendieck (Burnside) group.

Examples:

- Let U be a sub-(K, H)-biset of ${}_K G_H$. Then $ku = uh \Leftrightarrow k = {}^u h$, for $u \in U$, so U is a conjugation (K, H)-biset. In particular:

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Let $Conj^G_{K,H}$ denote the full subcategory of (K, H)-bisets consisting of conjugation (K, H)-bisets, and $\mathcal{B}^G_{K,H}$ its Grothendieck (Burnside) group.

Examples:

- Let U be a sub-(K, H)-biset of ${}_K G_H$. Then $ku = uh \Leftrightarrow k = {}^u h$, for $u \in U$, so U is a conjugation (K, H)-biset. In particular:
    - when $K \leq H$, the (K, H)-biset H is denoted by Res^H_K.

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Let $Conj_{K,H}^G$ denote the full subcategory of (K, H)-bisets consisting of conjugation (K, H)-bisets, and $\mathcal{B}_{K,H}^G$ its Grothendieck (Burnside) group.

Examples:

- Let U be a sub-(K, H)-biset of ${}_K G_H$. Then $ku = uh \Leftrightarrow k = {}^u h$, for $u \in U$, so U is a conjugation (K, H)-biset. In particular:
  - when $K \leq H$, the (K, H)-biset H is denoted by Res_K^H.
  - when $H \leq K$, the (K, H)-biset K is denoted by Ind_H^K.

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k,h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Let $Conj^G_{K,H}$ denote the full subcategory of (K, H)-bisets consisting of conjugation (K, H)-bisets, and $\mathcal{B}^G_{K,H}$ its Grothendieck (Burnside) group.

Examples:

- Let U be a sub-(K, H)-biset of ${}_K G_H$. Then $ku = uh \Leftrightarrow k = {}^u h$, for $u \in U$, so U is a conjugation (K, H)-biset. In particular:
  - when $K \leq H$, the (K, H)-biset H is denoted by Res^H_K.
  - when $H \leq K$, the (K, H)-biset K is denoted by Ind^K_H.
  - when $K = {}^g H$, for $g \in G$, the (K, H)-biset gH is denoted by $Cnj_{g,H}$.

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^{g}h.$$

Let $Conj^G_{K,H}$ denote the full subcategory of (K, H)-bisets consisting of conjugation (K, H)-bisets, and $\mathcal{B}^G_{K,H}$ its Grothendieck (Burnside) group.

Examples:

- Let U be a sub-(K, H)-biset of ${}_KG_H$. Then $ku = uh \Leftrightarrow k = {}^{u}h$, for $u \in U$, so U is a conjugation (K, H)-biset. In particular:
  - when $K \leq H$, the (K, H)-biset H is denoted by Res^H_K.
  - when $H \leq K$, the (K, H)-biset K is denoted by Ind^K_H.
  - when $K = {}^{g}H$, for $g \in G$, the (K, H)-biset gH is denoted by $Cnj_{g,H}$.
- When $L, K, H \leq G$, if V is a conjugation (L, K)-biset

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Let $Conj^G_{K,H}$ denote the full subcategory of (K, H)-bisets consisting of conjugation (K, H)-bisets, and $\mathcal{B}^G_{K,H}$ its Grothendieck (Burnside) group.

Examples:

- Let U be a sub-(K, H)-biset of ${}_K G_H$. Then $ku = uh \Leftrightarrow k = {}^u h$, for $u \in U$, so U is a conjugation (K, H)-biset. In particular:
  - when $K \leq H$, the (K, H)-biset H is denoted by Res^H_K.
  - when $H \leq K$, the (K, H)-biset K is denoted by Ind^K_H.
  - when $K = {}^g H$, for $g \in G$, the (K, H)-biset gH is denoted by $Cnj_{g,H}$.
- When $L, K, H \leq G$, if V is a conjugation (L, K)-biset and U is a conjugation (K, H)-biset

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^{g}h.$$

Let $Conj^{G}_{K,H}$ denote the full subcategory of (K, H)-bisets consisting of conjugation (K, H)-bisets, and $\mathcal{B}^{G}_{K,H}$ its Grothendieck (Burnside) group.

Examples:

- Let U be a sub-(K, H)-biset of ${}_{K}G_{H}$. Then $ku = uh \Leftrightarrow k = {}^{u}h$, for $u \in U$, so U is a conjugation (K, H)-biset. In particular:
  - when $K \leq H$, the (K, H)-biset H is denoted by Res^{H}_{K}.
  - when $H \leq K$, the (K, H)-biset K is denoted by Ind^{K}_{H}.
  - when $K = {}^{g}H$, for $g \in G$, the (K, H)-biset gH is denoted by $Cnj_{g,H}$.
- When $L, K, H \leq G$, if V is a conjugation (L, K)-biset and U is a conjugation (K, H)-biset, then $V \times_{H} U$ is a conjugation (L, H)-biset.

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Let $Conj^G_{K,H}$ denote the full subcategory of (K, H)-bisets consisting of conjugation (K, H)-bisets, and $\mathcal{B}^G_{K,H}$ its Grothendieck (Burnside) group.

Examples:

- Let U be a sub-(K, H)-biset of ${}_K G_H$. Then $ku = uh \Leftrightarrow k = {}^u h$, for $u \in U$, so U is a conjugation (K, H)-biset. In particular:
  - when $K \leq H$, the (K, H)-biset H is denoted by Res^H_K.
  - when $H \leq K$, the (K, H)-biset K is denoted by Ind^K_H.
  - when $K = {}^g H$, for $g \in G$, the (K, H)-biset gH is denoted by $Cnj_{g,H}$.
- When $L, K, H \leq G$, if V is a conjugation (L, K)-biset and U is a conjugation (K, H)-biset, then $V \times_H U$ is a conjugation (L, H)-biset.
- for $K, H \leq G$, any conjugation (K, H)-biset

Conjugation bisets

Fix a finite group G. When $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Let $Conj^G_{K,H}$ denote the full subcategory of (K, H)-bisets consisting of conjugation (K, H)-bisets, and $\mathcal{B}^G_{K,H}$ its Grothendieck (Burnside) group.

Examples:

- Let U be a sub-(K, H)-biset of ${}_K G_H$. Then $ku = uh \Leftrightarrow k = {}^u h$, for $u \in U$, so U is a conjugation (K, H)-biset. In particular:
  - when $K \leq H$, the (K, H)-biset H is denoted by Res^H_K.
  - when $H \leq K$, the (K, H)-biset K is denoted by Ind^K_H.
  - when $K = {}^g H$, for $g \in G$, the (K, H)-biset gH is denoted by $Cnj_{g,H}$.
- When $L, K, H \leq G$, if V is a conjugation (L, K)-biset and U is a conjugation (K, H)-biset, then $V \times_H U$ is a conjugation (L, H)-biset.
- for $K, H \leq G$, any conjugation (K, H)-biset is isomorphic to a disjoint union of conjugation bisets of the form $Ind^K_{{}^g A} \circ Cnj_{g,A} \circ Res^H_A$.

Mackey functors and biset functors

Fix a finite group G. Then:

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.

Mackey functors and biset functors

Fix a finite group G. Then:

- let “some” denote the subgroups of G.
- let “some” denote conjugation (K, H)-bisets, for $K, H \leq G$.

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$.

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K,H)-bisets, for $K,H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H) \\ \end{cases}$

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K,H)-bisets, for $K,H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

Mackey functors and biset functors

Fix a finite group G. Then:

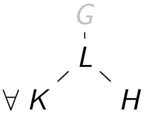
- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

In particular, the Mackey formula follows from



Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K,H)-bisets, for $K,H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

In particular, the Mackey formula follows from

$$\forall \begin{array}{ccc} & G & \\ & L & \\ K & & H \end{array}, \quad {}_KG_H = \bigsqcup_{g\in[K\backslash L/H]} KgH$$

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K,H)-bisets, for $K,H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

In particular, the Mackey formula follows from

$$\forall\ \begin{array}{ccc} & G & \\ & \vdots & \\ & L & \\ \diagup & & \diagdown \\ K & & H \end{array}, \quad {}_KG_H \cong \bigsqcup_{g\in[K\backslash L/H]}$$

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K,H)-bisets, for $K,H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

In particular, the Mackey formula follows from

$$\begin{array}{ccc} & G & \\ & \vert & \\ & L & \\ \forall\ K \diagup & & \diagdown H \end{array}, \quad {}_KG_H \cong \bigsqcup_{g\in[K\backslash L/H]} \qquad Res^H_{K^g\cap H}$$

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K,H)-bisets, for $K,H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2.
   - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

In particular, the Mackey formula follows from

$$\forall \begin{tikzcd}[column sep=small] & G \arrow[d, dash, dashed] & \\ & L \arrow[dl, dash] \arrow[dr, dash] & \\ K & & H \end{tikzcd}, \quad {}_KG_H \cong \bigsqcup_{g\in[K\backslash L/H]} Cnj_{g,K^g\cap H} \circ Res^H_{K^g\cap H}$$

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

In particular, the Mackey formula follows from

$$\forall \begin{tikzcd}[column sep=small] & G \arrow[d, dash, dashed] & \\ & L \arrow[dl, dash] \arrow[dr, dash] & \\ K & & H \end{tikzcd}, \quad {}_KG_H \cong \bigsqcup_{g \in [K \backslash L / H]} Ind_{K \cap {}^gH}^K \circ Cnj_{g, K^g \cap H} \circ Res_{K^g \cap H}^H.$$

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K,H)-bisets, for $K,H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2.
   - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

Question:

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2.
   - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

Question: Let M be a Mackey functor for G.

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

Question: Let M be a Mackey functor for G. Is there a biset functor F such that $M \cong M_F$?

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K,H)-bisets, for $K,H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

Question: Let M be a Mackey functor for G. Is there a biset functor F such that $M \cong M_F$?

Answer: No in general.

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2.
   - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

Question: Let M be a Mackey functor for G. Is there a biset functor F such that $M \cong M_F$?

Answer: No in general. If $g \in C_G(H)$

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

Question: Let M be a Mackey functor for G. Is there a biset functor F such that $M \cong M_F$?

Answer: No in general. If $g \in C_G(H)$, then $gH \cong H$ as (H, H)-biset

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K,H)-bisets, for $K,H\leq G$.

Set $\mathcal{F}_G=\mathcal{F}_{some,some}$. If $F\in\mathcal{F}_G$:

1. for any $H\leq G$, set $M_F(H)=F(H)$.
2.
   - for $H\leq K\leq G$, define $\begin{cases} r_H^K=F(Res_H^K):M_F(K)\to M_F(H),\\ t_H^K=F(Ind_H^K):M_F(H)\to M_F(K).\end{cases}$
   - for $H\leq G$ and $g\in G$, define $c_{g,H}=F(Cnj_{g,H}):M_F(H)\to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

Question: Let M be a Mackey functor for G. Is there a biset functor F such that $M\cong M_F$?

Answer: No in general. If $g\in C_G(H)$, then $gH\cong H$ as (H,H)-biset, hence $Cnj_{g,H}\cong Cnj_{1,H}\cong Id_H$.

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K,H)-bisets, for $K,H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

Question: Let M be a Mackey functor for G. Is there a biset functor F such that $M \cong M_F$?
Answer: No in general. If $g \in C_G(H)$, then $gH \cong H$ as (H,H)-biset, hence $Cnj_{g,H} \cong Cnj_{1,H} \cong Id_H$. Hence $c_{g,H} = Id_{M_F(H)}$

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K,H)-bisets, for $K,H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

Question: Let M be a Mackey functor for G. Is there a biset functor F such that $M \cong M_F$?

Answer: No in general. If $g \in C_G(H)$, then $gH \cong H$ as (H,H)-biset, hence $Cnj_{g,H} \cong Cnj_{1,H} \cong Id_H$. Hence $c_{g,H} = Id_{M_F(H)}$, and $C_G(H)$ acts trivially on $M_F(H)$, for any $H \leq G$.

Mackey functors and biset functors

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K, H)-bisets, for $K, H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

Question: Let M be a Mackey functor for G. Is there a biset functor F such that $M \cong M_F$?

Answer: No in general. If $g \in C_G(H)$, then $gH \cong H$ as (H, H)-biset, hence $Cnj_{g,H} \cong Cnj_{1,H} \cong Id_H$. Hence $c_{g,H} = Id_{M_F(H)}$, and $C_G(H)$ acts trivially on $M_F(H)$, for any $H \leq G$.

Fix a finite group G. Then:

- let "some" denote the subgroups of G.
- let "some" denote conjugation (K,H)-bisets, for $K,H \leq G$.

Set $\mathcal{F}_G = \mathcal{F}_{some,some}$. If $F \in \mathcal{F}_G$:

1. for any $H \leq G$, set $M_F(H) = F(H)$.
2. - for $H \leq K \leq G$, define $\begin{cases} r_H^K = F(Res_H^K) : M_F(K) \to M_F(H), \\ t_H^K = F(Ind_H^K) : M_F(H) \to M_F(K). \end{cases}$
   - for $H \leq G$ and $g \in G$, define $c_{g,H} = F(Cnj_{g,H}) : M_F(H) \to M_F({}^gH)$.

Then M_F is a Mackey functor for G.

Question: Let M be a Mackey functor for G. Is there a biset functor F such that $M \cong M_F$?

Answer: No in general. If $g \in C_G(H)$, then $gH \cong H$ as (H,H)-biset, hence $Cnj_{g,H} \cong Cnj_{1,H} \cong Id_H$. Hence $c_{g,H} = Id_{M_F(H)}$, and $C_G(H)$ acts trivially on $M_F(H)$, for any $H \leq G$.

Conjugation invariant Mackey functors

Conjugation invariant Mackey functors

Definition (Hambleton-Taylor-Williams 2010)

Conjugation invariant Mackey functors

Definition (Hambleton-Taylor-Williams 2010)

Let G be a finite group.

Conjugation invariant Mackey functors

Definition (Hambleton-Taylor-Williams 2010)

Let G be a finite group. A Mackey functor M for G is said to be conjugation invariant

Conjugation invariant Mackey functors

Definition (Hambleton-Taylor-Williams 2010)

Let G be a finite group. A Mackey functor M for G is said to be conjugation invariant if $C_G(H)$ acts trivially on $M(H)$, for any $H \leq G$.

Conjugation invariant Mackey functors

Definition (Hambleton-Taylor-Williams 2010)

Let G be a finite group. A Mackey functor M for G is said to be conjugation invariant if $C_G(H)$ acts trivially on $M(H)$, for any $H \leq G$.

Let $Mack^c(G)$ denote the subcategory of $Mack(G)$ consisting of conjugation invariant Mackey functors.

Conjugation invariant Mackey functors

Definition (Hambleton-Taylor-Williams 2010)

Let G be a finite group. A Mackey functor M for G is said to be conjugation invariant if $C_G(H)$ acts trivially on $M(H)$, for any $H \leq G$.

Let $Mack^c(G)$ denote the subcategory of $Mack(G)$ consisting of conjugation invariant Mackey functors.
Recall that $\mathcal{F}_G$ denotes the category of biset functors defined on subgroups of G, with conjugation (K, H)-bisets as morphisms.

Conjugation invariant Mackey functors

Definition (Hambleton-Taylor-Williams 2010)

Let G be a finite group. A Mackey functor M for G is said to be conjugation invariant if $C_G(H)$ acts trivially on $M(H)$, for any $H \leq G$.

Let $Mack^c(G)$ denote the subcategory of $Mack(G)$ consisting of conjugation invariant Mackey functors.
Recall that $\mathcal{F}_G$ denotes the category of biset functors defined on subgroups of G, with conjugation (K, H)-bisets as morphisms.

Theorem (Hambleton-Taylor-Williams 2010)

The functor $F \mapsto M_F$ is an equivalence of categories $\mathcal{F}_G \to Mack^c(G)$.

Mackey functors revisited (Dress)

Mackey functors revisited

Let G be a finite group.

Mackey functors revisited

Let G be a finite group. Let G-**set** denote the category of finite G-sets.

Mackey functors revisited

Let G be a finite group. Let G-**set** denote the category of finite G-sets. Let $\mathbb{Z}$-**Mod** denote the category of abelian groups.

Mackey functors revisited

Let G be a finite group. Let G-**set** denote the category of finite G-sets. Let $\mathbb{Z}$-**Mod** denote the category of abelian groups.

Definition (Dress)

A Mackey functor M for G is a bivariant functor (M^*, M_*) from G-**set** to $\mathbb{Z}$-**Mod**, i.e.

Mackey functors revisited

Let G be a finite group. Let G-**set** denote the category of finite G-sets. Let $\mathbb{Z}$-**Mod** denote the category of abelian groups.

Definition (Dress)

A Mackey functor M for G is a bivariant functor (M^*, M_*) from G-**set** to $\mathbb{Z}$-**Mod**, i.e.

- M^* is a contravariant functor G-**set**$\rightarrow$ $\mathbb{Z}$-**Mod**.

Mackey functors revisited

Let G be a finite group. Let G-**set** denote the category of finite G-sets. Let $\mathbb{Z}$-**Mod** denote the category of abelian groups.

Definition (Dress)

A Mackey functor M for G is a bivariant functor (M^*, M_*) from G-**set** to $\mathbb{Z}$-**Mod**, i.e.

- M^* is a contravariant functor G-**set**$\to$ $\mathbb{Z}$-**Mod**.
- M_* is a covariant functor G-**set**$\to$ $\mathbb{Z}$-**Mod**.

Mackey functors revisited

Let G be a finite group. Let G-**set** denote the category of finite G-sets. Let $\mathbb{Z}$-**Mod** denote the category of abelian groups.

Definition (Dress)

A Mackey functor M for G is a bivariant functor (M^*, M_*) from G-**set** to $\mathbb{Z}$-**Mod**, i.e.

- M^* is a contravariant functor G-**set**$\to$ $\mathbb{Z}$-**Mod**.
- M_* is a covariant functor G-**set**$\to$ $\mathbb{Z}$-**Mod**.
- $M^*(X) = M_*(X)$ for any finite G-set X.

Mackey functors revisited

Let G be a finite group. Let G-**set** denote the category of finite G-sets. Let $\mathbb{Z}$-**Mod** denote the category of abelian groups.

Definition (Dress)

A Mackey functor M for G is a bivariant functor (M^*, M_*) from G-**set** to $\mathbb{Z}$-**Mod**, such that

Mackey functors revisited

Let G be a finite group. Let G-**set** denote the category of finite G-sets. Let $\mathbb{Z}$-**Mod** denote the category of abelian groups.

Definition (Dress)

A Mackey functor M for G is a bivariant functor (M^*, M_*) from G-**set** to $\mathbb{Z}$-**Mod**, such that

1. for any finite G-sets X and Y, the maps

$$M(X \sqcup Y) \underset{(M_*(i_X),M_*(i_Y))}{\overset{(M^*(i_X),M^*(i_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$

Mackey functors revisited

Let G be a finite group. Let G-**set** denote the category of finite G-sets. Let $\mathbb{Z}$-**Mod** denote the category of abelian groups.

Definition (Dress)

A Mackey functor M for G is a bivariant functor (M^*, M_*) from G-**set** to $\mathbb{Z}$-**Mod**, such that

1. for any finite G-sets X and Y, the maps

$$M(X \sqcup Y) \underset{(M_*(i_X), M_*(i_Y))}{\overset{(M^*(i_X), M^*(i_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$

   are inverse to each other

Mackey functors revisited

Let G be a finite group. Let G-**set** denote the category of finite G-sets. Let $\mathbb{Z}$-**Mod** denote the category of abelian groups.

Definition (Dress)

A Mackey functor M for G is a bivariant functor (M^*, M_*) from G-**set** to $\mathbb{Z}$-**Mod**, such that

1. for any finite G-sets X and Y, the maps

$$M(X \sqcup Y) \underset{(M_*(i_X),M_*(i_Y))}{\overset{(M^*(i_X),M^*(i_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$

are inverse to each other, where $X \overset{i_X}{\hookrightarrow} X \sqcup Y \overset{i_Y}{\hookleftarrow} Y$.

Mackey functors revisited

Let G be a finite group. Let G-**set** denote the category of finite G-sets. Let $\mathbb{Z}$-**Mod** denote the category of abelian groups.

Definition (Dress)

A Mackey functor M for G is a bivariant functor (M^*, M_*) from G-**set** to $\mathbb{Z}$-**Mod**, such that

1. for any finite G-sets X and Y, the maps
$$M(X \sqcup Y) \underset{(M_*(i_X), M_*(i_Y))}{\overset{(M^*(i_X), M^*(i_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$
are inverse to each other, where $X \overset{i_X}{\hookrightarrow} X \sqcup Y \overset{i_Y}{\hookleftarrow} Y$.
2. if $\begin{array}{ccc} X & \overset{a}{\to} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ Z & \underset{d}{\to} & T \end{array}$ is a cartesian (i.e. pullback) square of finite G-sets

Mackey functors revisited

Let G be a finite group. Let G-**set** denote the category of finite G-sets. Let $\mathbb{Z}$-**Mod** denote the category of abelian groups.

Definition (Dress)

A Mackey functor M for G is a bivariant functor (M^*, M_*) from G-**set** to $\mathbb{Z}$-**Mod**, such that

1. for any finite G-sets X and Y, the maps
$$M(X \sqcup Y) \underset{(M_*(i_X), M_*(i_Y))}{\overset{(M^*(i_X), M^*(i_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$
are inverse to each other, where $X \xhookrightarrow{i_X} X \sqcup Y \xhookleftarrow{i_Y} Y$.
2. if $\begin{array}{ccc} X & \xrightarrow{a} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ Z & \xrightarrow[d]{} & T \end{array}$ is a cartesian (i.e. pullback) square of finite G-sets, then
$$M^*(d) \circ M_*(c) = M_*(b) \circ M^*(a).$$

Mackey functors revisited

Let G be a finite group. Let G-**set** denote the category of finite G-sets. Let $\mathbb{Z}$-**Mod** denote the category of abelian groups.

Definition (Dress)

A Mackey functor M for G is a bivariant functor (M^*, M_*) from G-**set** to $\mathbb{Z}$-**Mod**, such that

1. for any finite G-sets X and Y, the maps
$$M(X \sqcup Y) \underset{(M_*(i_X), M_*(i_Y))}{\overset{(M^*(i_X), M^*(i_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$
are inverse to each other, where $X \overset{i_X}{\hookrightarrow} X \sqcup Y \overset{i_Y}{\hookleftarrow} Y$.

2. if $\begin{array}{ccc} X & \overset{a}{\to} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ Z & \underset{d}{\to} & T \end{array}$ is a cartesian (i.e. pullback) square of finite G-sets, then
$$M^*(d) \circ M_*(c) = M_*(b) \circ M^*(a).$$

Mackey functors revisited (Lindner)

Mackey functors revisited

Let $\mathcal{S}_G$ denote the following category (of spans of G-sets):

Mackey functors revisited

Let $\mathcal{S}_G$ denote the following category:

- the objects of $\mathcal{S}_G$ are the finite G-sets.

Mackey functors revisited

Let $\mathcal{S}_G$ denote the following category:

- the objects of $\mathcal{S}_G$ are the finite G-sets.
- $Hom_{\mathcal{S}_G}(X, Y) = \mathcal{B}\big({}_G(Y \times X)\big)$ for finite G-sets X and Y

Mackey functors revisited

Let $\mathcal{S}_G$ denote the following category:

- the objects of $\mathcal{S}_G$ are the finite G-sets.
- $Hom_{\mathcal{S}_G}(X, Y) = \mathcal{B}\big({}_G(Y \times X)\big)$ for finite G-sets X and Y, where $\mathcal{B}\big({}_G(Y \times X)\big)$ is the Grothendieck group (Burnside group) of the category G-$\mathbf{set}_{\downarrow Y \times X}$.

Mackey functors revisited

Let $\mathcal{S}_G$ denote the following category:

- the objects of $\mathcal{S}_G$ are the finite G-sets.
- $Hom_{\mathcal{S}_G}(X, Y) = \mathcal{B}\big({}_G(Y \times X)\big)$ for finite G-sets X and Y, where $\mathcal{B}\big({}_G(Y \times X)\big)$ is the Grothendieck group (Burnside group) of the category G-$\mathbf{set}_{\downarrow Y \times X}$.
- composition is induced by pullback

Mackey functors revisited

Let $\mathcal{S}_G$ denote the following category:

- the objects of $\mathcal{S}_G$ are the finite G-sets.
- $Hom_{\mathcal{S}_G}(X, Y) = \mathcal{B}\big({}_G(Y \times X)\big)$ for finite G-sets X and Y, where $\mathcal{B}\big({}_G(Y \times X)\big)$ is the Grothendieck group (Burnside group) of the category G-$\mathbf{set}_{\downarrow Y \times X}$.
- composition is induced by pullback, from

$$\begin{tikzcd} & D \arrow[dl] \arrow[dr] & \\ Z & & Y \end{tikzcd} \circ \begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd} = \begin{tikzcd} & & E \arrow[dl] \arrow[dr] & & \\ & D \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\boxed{\mathrm{PB}}"] & & C \arrow[dl] \arrow[dr] & \\ Z & & Y & & X \end{tikzcd} .$$

Let $\mathcal{S}_G$ denote the following category:

- the objects of $\mathcal{S}_G$ are the finite G-sets.
- $Hom_{\mathcal{S}_G}(X,Y) = \mathcal{B}\big({}_G(Y\times X)\big)$ for finite G-sets X and Y, where $\mathcal{B}\big({}_G(Y\times X)\big)$ is the Grothendieck group (Burnside group) of the category G-$\mathbf{set}_{\downarrow Y\times X}$.
- composition is induced by pullback, from

$$\begin{tikzcd}[column sep=small] & D \arrow[dl] \arrow[dr] & \\ Z & & Y \end{tikzcd} \circ \begin{tikzcd}[column sep=small] & C \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd} = \begin{tikzcd}[column sep=small] & & E \arrow[dl] \arrow[dr] & & \\ & D \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\boxed{\mathrm{PB}}"] & & C \arrow[dl] \arrow[dr] & \\ Z & & Y & & X \end{tikzcd} .$$

- the identity morphism of X is $\begin{tikzcd}[column sep=small] & X \arrow[dl, equal] \arrow[dr, equal] & \\ X & & X \end{tikzcd}$.

Mackey functors revisited

Let $\mathcal{S}_G$ denote the following category:

- the objects of $\mathcal{S}_G$ are the finite G-sets.
- $Hom_{\mathcal{S}_G}(X,Y) = \mathcal{B}\big({}_G(Y \times X)\big)$ for finite G-sets X and Y, where $\mathcal{B}\big({}_G(Y \times X)\big)$ is the Grothendieck group (Burnside group) of the category G-$\mathbf{set}_{\downarrow Y \times X}$.
- composition is induced by pullback, from

$$\begin{tikzcd}[column sep=small] & D \arrow[dl] \arrow[dr] & \\ Z & & Y \end{tikzcd} \circ \begin{tikzcd}[column sep=small] & C \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd} = \begin{tikzcd}[column sep=small] & & E \arrow[dl] \arrow[dr] & & \\ & D \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\boxed{\text{PB}}"] & & C \arrow[dl] \arrow[dr] & \\ Z & & Y & & X \end{tikzcd} .$$

- the identity morphism of X is $\begin{tikzcd}[column sep=small] & X \arrow[dl, equal] \arrow[dr, equal] & \\ X & & X \end{tikzcd}$.

Then a Mackey functor for G

Mackey functors revisited

Let $\mathcal{S}_G$ denote the following category:

- the objects of $\mathcal{S}_G$ are the finite G-sets.
- $Hom_{\mathcal{S}_G}(X,Y) = \mathcal{B}\big({}_G(Y \times X)\big)$ for finite G-sets X and Y, where $\mathcal{B}\big({}_G(Y \times X)\big)$ is the Grothendieck group (Burnside group) of the category $G\text{-}\mathbf{set}_{\downarrow Y \times X}$.
- composition is induced by pullback, from

$$\begin{tikzcd}[column sep=small] & D \arrow[dl] \arrow[dr] & \\ Z & & Y \end{tikzcd} \circ \begin{tikzcd}[column sep=small] & C \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd} = \begin{tikzcd}[column sep=small] & & E \arrow[dl] \arrow[dr] & & \\ & D \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\boxed{\text{PB}}"] & & C \arrow[dl] \arrow[dr] & \\ Z & & Y & & X \end{tikzcd} .$$

- the identity morphism of X is $\begin{tikzcd}[column sep=small] & X \arrow[dl, equal] \arrow[dr, equal] & \\ X & & X \end{tikzcd}$.

Then a Mackey functor for G is an additive functor $\mathcal{S}_G \to \mathbb{Z}\text{-}\mathbf{Mod}$.

Biset functors revisited

Biset functors revisited

Let $\mathcal{C}_G$ denote the following category:

Biset functors revisited

Let $\mathcal{C}_G$ denote the following category:

- the objects of $\mathcal{C}_G$ are the subgroups of G.

Biset functors revisited

Let $\mathcal{C}_G$ denote the following category:

- the objects of $\mathcal{C}_G$ are the subgroups of G.
- if $H, K \leq G$, then $Hom_{\mathcal{C}_G}(H, K)$ is the Grothendieck group $\mathcal{B}^G_{K,H}$

Biset functors revisited

Let $\mathcal{C}_G$ denote the following category:

- the objects of $\mathcal{C}_G$ are the subgroups of G.
- if $H, K \leq G$, then $Hom_{\mathcal{C}_G}(H, K)$ is the Grothendieck group $\mathcal{B}^G_{K,H}$ of the category $Conj^G_{K,H}$.

Biset functors revisited

Let $\mathcal{C}_G$ denote the following category:

- the objects of $\mathcal{C}_G$ are the subgroups of G.
- if $H, K \leq G$, then $Hom_{\mathcal{C}_G}(H, K)$ is the Grothendieck group $\mathcal{B}^G_{K,H}$ of the category $Conj^G_{K,H}$.
- composition in $\mathcal{C}_G$ is induced by the composition of bisets.

Biset functors revisited

Let $\mathcal{C}_G$ denote the following category:

- the objects of $\mathcal{C}_G$ are the subgroups of G.
- if $H, K \leq G$, then $Hom_{\mathcal{C}_G}(H, K)$ is the Grothendieck group $\mathcal{B}^G_{K,H}$ of the category $Conj^G_{K,H}$.
- composition in $\mathcal{C}_G$ is induced by the composition of bisets.
- the identity morphism of $H \leq G$ is the (H, H)-biset H.

Biset functors revisited

Let $\mathcal{C}_G$ denote the following category:

- the objects of $\mathcal{C}_G$ are the subgroups of G.
- if $H, K \leq G$, then $Hom_{\mathcal{C}_G}(H, K)$ is the Grothendieck group $\mathcal{B}^G_{K,H}$ of the category $Conj^G_{K,H}$.
- composition in $\mathcal{C}_G$ is induced by the composition of bisets.
- the identity morphism of $H \leq G$ is the (H, H)-biset H.

Then the category $\mathcal{F}_G$ is equivalent to the category of additive functors from $\mathcal{C}_G$ to $\mathbb{Z}$-**Mod**.

Conjugation bisets revisited

Conjugation bisets revisited

Recall that when $H, K \leq G$, a conjugation (K, H)-biset U

Conjugation bisets revisited

Recall that when $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$

Conjugation bisets revisited

Recall that when $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^{g}h.$$

Conjugation bisets revisited

Recall that when $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Equivalently

Conjugation bisets revisited

Recall that when $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Equivalently U is a (K, H)-biset such that there exists a morphism of (K, H)-bisets $U \to {}_K G_H$

Conjugation bisets revisited

Recall that when $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Equivalently U is a (K, H)-biset such that there exists a morphism of (K, H)-bisets $U \to {}_K G_H$:

$\boxed{\rightarrow}$ if U is a conjugation (K, H)-biset, then for each $u \in [K \backslash U / H]$, choose g_u as above

Conjugation bisets revisited

Recall that when $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^{g}h.$$

Equivalently U is a (K, H)-biset such that there exists a morphism of (K, H)-bisets $U \to {}_K G_H$:

$\boxed{\rightarrow}$ if U is a conjugation (K, H)-biset, then for each $u \in [K \backslash U/H]$, choose g_u as above, and define a map $\alpha : U \to G$ by $\alpha(kuh) = kg_u h$, for $u \in [K \backslash U/H]$, $k \in K$, $h \in H$.

Conjugation bisets revisited

Recall that when $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Equivalently U is a (K, H)-biset such that there exists a morphism of (K, H)-bisets $U \to {}_K G_H$:

$\boxed{\rightarrow}$ if U is a conjugation (K, H)-biset, then for each $u \in [K \backslash U / H]$, choose g_u as above, and define a map $\alpha : U \to G$ by $\alpha(kuh) = kg_u h$, for $u \in [K \backslash U / H]$, $k \in K$, $h \in H$.

$\boxed{\leftarrow}$ if $\beta : U \to G$ is a map of (K, H)-bisets

Conjugation bisets revisited

Recall that when $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Equivalently U is a (K, H)-biset such that there exists a morphism of (K, H)-bisets $U \to {}_K G_H$:

$\boxed{\rightarrow}$ if U is a conjugation (K, H)-biset, then for each $u \in [K \backslash U / H]$, choose g_u as above, and define a map $\alpha : U \to G$ by $\alpha(kuh) = kg_u h$, for $u \in [K \backslash U / H]$, $k \in K$, $h \in H$.

$\boxed{\leftarrow}$ if $\beta : U \to G$ is a map of (K, H)-bisets, then for any $u \in U$

$$\forall (k, h) \in K \times H, \quad ku = uh$$

Conjugation bisets revisited

Recall that when $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^{g}h.$$

Equivalently U is a (K, H)-biset such that there exists a morphism of (K, H)-bisets $U \to {}_K G_H$:

$\boxed{\rightarrow}$ if U is a conjugation (K, H)-biset, then for each $u \in [K \backslash U / H]$, choose g_u as above, and define a map $\alpha : U \to G$ by $\alpha(kuh) = kg_u h$, for $u \in [K \backslash U / H]$, $k \in K$, $h \in H$.

$\boxed{\leftarrow}$ if $\beta : U \to G$ is a map of (K, H)-bisets, then for any $u \in U$

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k\beta(u) = \beta(u)h$$

Conjugation bisets revisited

Recall that when $H, K \leq G$, a conjugation (K, H)-biset U is a biset such that for each $u \in U$, there exists $g = g_u \in G$ such that

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k = {}^g h.$$

Equivalently U is a (K, H)-biset such that there exists a morphism of (K, H)-bisets $U \to {}_K G_H$:

$\boxed{\rightarrow}$ if U is a conjugation (K, H)-biset, then for each $u \in [K \backslash U / H]$, choose g_u as above, and define a map $\alpha : U \to G$ by $\alpha(kuh) = kg_u h$, for $u \in [K \backslash U / H]$, $k \in K$, $h \in H$.

$\boxed{\leftarrow}$ if $\beta : U \to G$ is a map of (K, H)-bisets, then for any $u \in U$

$$\forall (k, h) \in K \times H, \quad ku = uh \Rightarrow k\beta(u) = \beta(u)h \Rightarrow k = {}^g h, \text{ for } g = \beta(u).$$

Let $K, H \leq G$.

Let $K, H \leq G$. There is an obvious forgetful functor

$$\begin{pmatrix} U \\ \big\downarrow{\scriptstyle \alpha} \\ {}_K G_H \end{pmatrix} \in (K,H)\text{-}\mathbf{biset}_{\downarrow {}_K G_H}$$

Let $K, H \leq G$. There is an obvious forgetful functor

$$\begin{pmatrix} U \\ \downarrow \alpha \\ {}_K G_H \end{pmatrix} \in (K,H)\text{-}\mathbf{biset}_{\downarrow {}_K G_H} \mapsto U \in Conj_{K,H}^G$$

Let $K, H \leq G$. There is an obvious forgetful functor

$$\begin{pmatrix} U \\ \downarrow^{\alpha} \\ {}_K G_H \end{pmatrix} \in (K, H)\text{-}\mathbf{biset}_{\downarrow {}_K G_H} \mapsto U \in Conj_{K,H}^G$$

which induces a surjective group homomorphism

$$\mathcal{B}({}_K G_H) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{B}_{K,H}^G.$$

Let $K, H \leq G$. There is an obvious forgetful functor

$$\begin{pmatrix} U \\ \downarrow^{\alpha} \\ {}_K G_H \end{pmatrix} \in (K,H)\text{-}\mathbf{biset}_{\downarrow_K G_H} \mapsto U \in Conj^G_{K,H}$$

which induces a surjective group homomorphism

$$\mathcal{B}({}_K G_H) \twoheadrightarrow \mathcal{B}^G_{K,H}.$$

This is compatible with the composition

Let $K, H \leq G$. There is an obvious forgetful functor

$$\begin{pmatrix} U \\ \downarrow\alpha \\ {}_K G_H \end{pmatrix} \in (K,H)\text{-}\mathbf{biset}_{\downarrow_K G_H} \mapsto U \in Conj_{K,H}^G$$

which induces a surjective group homomorphism

$$\mathcal{B}({}_K G_H) \twoheadrightarrow \mathcal{B}_{K,H}^G.$$

This is compatible with the composition, if

$$\begin{pmatrix} V \\ \downarrow\beta \\ {}_L G_K \end{pmatrix} \circ \begin{pmatrix} U \\ \downarrow\alpha \\ {}_K G_H \end{pmatrix} =$$

Let $K, H \leq G$. There is an obvious forgetful functor

$$\begin{pmatrix} U \\ \big\downarrow{\scriptstyle\alpha} \\ {}_K G_H \end{pmatrix} \in (K,H)\text{-}\mathbf{biset}_{\downarrow_K G_H} \mapsto U \in Conj^G_{K,H}$$

which induces a surjective group homomorphism

$$\mathcal{B}({}_K G_H) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{B}^G_{K,H}.$$

This is compatible with the composition, if

$$\begin{pmatrix} V \\ \big\downarrow{\scriptstyle\beta} \\ {}_L G_K \end{pmatrix} \circ \begin{pmatrix} U \\ \big\downarrow{\scriptstyle\alpha} \\ {}_K G_H \end{pmatrix} = \begin{pmatrix} V \times_K U \\ \big\downarrow{\scriptstyle\beta\times_K\alpha} \\ G \times_K G \\ \big\downarrow{\scriptstyle\mu} \\ G \end{pmatrix}$$

From subgroups to G-sets

From subgroups to G-sets

Moreover ${}_KG_H$ is the restriction to (K,H) of the (G,G)-biset G.

From subgroups to G-sets

Moreover ${}_K G_H$ is the restriction to (K, H) of the (G, G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ.

From subgroups to G-sets

Moreover ${}_K G_H$ is the restriction to (K, H) of the (G, G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set

Moreover ${}_K G_H$ is the restriction to (K, H) of the (G, G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set, and let $Y = Ind_\Lambda^\Gamma X = \Gamma \times_\Lambda X$

From subgroups to G-sets

Moreover ${}_K G_H$ is the restriction to (K, H) of the (G, G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set, and let $Y = Ind_\Lambda^\Gamma X$.

From subgroups to G-sets

Moreover ${}_K G_H$ is the restriction to (K, H) of the (G, G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set, and let $Y = \mathrm{Ind}_\Lambda^\Gamma X$. Then the categories Λ-$\mathbf{set}_{\downarrow X}$ and Γ-$\mathbf{set}_{\downarrow Y}$ are equivalent

From subgroups to G-sets

Moreover ${}_K G_H$ is the restriction to (K, H) of the (G, G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set, and let $Y = Ind_\Lambda^\Gamma X$. Then the categories Λ-$\mathbf{set}_{\downarrow X}$ and Γ-$\mathbf{set}_{\downarrow Y}$ are equivalent, and in particular

$$\mathcal{B}({}_\Gamma Y) \cong \mathcal{B}({}_\Lambda X).$$

From subgroups to G-sets

Moreover ${}_KG_H$ is the restriction to (K,H) of the (G,G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set, and let $Y = Ind_\Lambda^\Gamma X$. Then the categories Λ-$\mathbf{set}_{\downarrow X}$ and Γ-$\mathbf{set}_{\downarrow Y}$ are equivalent, and in particular

$$\mathcal{B}({}_\Gamma Y) \cong \mathcal{B}({}_\Lambda X).$$

In particular if $X = Res_\Lambda^\Gamma Z$ for some finite Γ-set Z

From subgroups to G-sets

Moreover ${}_K G_H$ is the restriction to (K, H) of the (G, G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set, and let $Y = Ind_\Lambda^\Gamma X$. Then the categories Λ-$\mathbf{set}_{\downarrow X}$ and Γ-$\mathbf{set}_{\downarrow Y}$ are equivalent, and in particular

$$\mathcal{B}({}_\Gamma Y) \cong \mathcal{B}({}_\Lambda X).$$

In particular if $X = Res_\Lambda^\Gamma Z$ for some finite Γ-set Z, then

$$\mathcal{B}\big({}_\Lambda(Res_\Lambda^\Gamma Z)\big) \cong$$

From subgroups to G-sets

Moreover ${}_K G_H$ is the restriction to (K, H) of the (G, G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set, and let $Y = Ind_\Lambda^\Gamma X$. Then the categories Λ-$\mathbf{set}_{\downarrow X}$ and Γ-$\mathbf{set}_{\downarrow Y}$ are equivalent, and in particular

$$\mathcal{B}({}_\Gamma Y) \cong \mathcal{B}({}_\Lambda X).$$

In particular if $X = Res_\Lambda^\Gamma Z$ for some finite Γ-set Z, then

$$\mathcal{B}\big({}_\Lambda(Res_\Lambda^\Gamma Z)\big) \cong \mathcal{B}\big({}_\Gamma(Ind_\Lambda^\Gamma Res_\Lambda^\Gamma Z)\big) \cong$$

From subgroups to G-sets

Moreover ${}_K G_H$ is the restriction to (K,H) of the (G,G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set, and let $Y = Ind_\Lambda^\Gamma X$. Then the categories Λ-$\mathbf{set}_{\downarrow X}$ and Γ-$\mathbf{set}_{\downarrow Y}$ are equivalent, and in particular

$$\mathcal{B}({}_\Gamma Y) \cong \mathcal{B}({}_\Lambda X).$$

In particular if $X = Res_\Lambda^\Gamma Z$ for some finite Γ-set Z, then

$$\mathcal{B}\big({}_\Lambda(Res_\Lambda^\Gamma Z)\big) \cong \mathcal{B}\big({}_\Gamma(Ind_\Lambda^\Gamma Res_\Lambda^\Gamma Z)\big) \cong \mathcal{B}\Big({}_\Gamma\big((\Gamma/\Lambda) \times Z\big)\Big).$$

From subgroups to G-sets

Moreover ${}_K G_H$ is the restriction to (K, H) of the (G, G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set, and let $Y = Ind_\Lambda^\Gamma X$. Then the categories Λ-$\mathbf{set}_{\downarrow X}$ and Γ-$\mathbf{set}_{\downarrow Y}$ are equivalent, and in particular

$$\mathcal{B}({}_\Gamma Y) \cong \mathcal{B}({}_\Lambda X).$$

In particular if $X = Res_\Lambda^\Gamma Z$ for some finite Γ-set Z, then

$$\mathcal{B}\big({}_\Lambda(Res_\Lambda^\Gamma Z)\big) \cong \mathcal{B}\big({}_\Gamma(Ind_\Lambda^\Gamma Res_\Lambda^\Gamma Z)\big) \cong \mathcal{B}\Big({}_\Gamma\big((\Gamma/\Lambda) \times Z\big)\Big).$$

For $\Gamma = G \times G^{\mathrm{op}}$

From subgroups to G-sets

Moreover ${}_KG_H$ is the restriction to (K,H) of the (G,G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set, and let $Y = Ind_\Lambda^\Gamma X$. Then the categories Λ-$\mathbf{set}_{\downarrow X}$ and Γ-$\mathbf{set}_{\downarrow Y}$ are equivalent, and in particular

$$\mathcal{B}({}_\Gamma Y) \cong \mathcal{B}({}_\Lambda X).$$

In particular if $X = Res_\Lambda^\Gamma Z$ for some finite Γ-set Z, then

$$\mathcal{B}\big({}_\Lambda(Res_\Lambda^\Gamma Z)\big) \cong \mathcal{B}\big({}_\Gamma(Ind_\Lambda^\Gamma Res_\Lambda^\Gamma Z)\big) \cong \mathcal{B}\Big({}_\Gamma\big((\Gamma/\Lambda)\times Z\big)\Big).$$

For $\Gamma = G \times G^{\mathrm{op}}$, $\Lambda = K \times H^{\mathrm{op}}$

From subgroups to G-sets

Moreover ${}_K G_H$ is the restriction to (K,H) of the (G,G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set, and let $Y = Ind_\Lambda^\Gamma X$. Then the categories Λ-$\mathbf{set}_{\downarrow X}$ and Γ-$\mathbf{set}_{\downarrow Y}$ are equivalent, and in particular

$$\mathcal{B}({}_\Gamma Y) \cong \mathcal{B}({}_\Lambda X).$$

In particular if $X = Res_\Lambda^\Gamma Z$ for some finite Γ-set Z, then

$$\mathcal{B}\big({}_\Lambda(Res_\Lambda^\Gamma Z)\big) \cong \mathcal{B}\big({}_\Gamma(Ind_\Lambda^\Gamma Res_\Lambda^\Gamma Z)\big) \cong \mathcal{B}\Big({}_\Gamma\big((\Gamma/\Lambda)\times Z\big)\Big).$$

For $\Gamma = G \times G^{\mathrm{op}}$, $\Lambda = K \times H^{\mathrm{op}}$, and $Z = {}_G G_G$

From subgroups to G-sets

Moreover ${}_K G_H$ is the restriction to (K, H) of the (G, G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set, and let $Y = Ind_\Lambda^\Gamma X$. Then the categories Λ-$\mathbf{set}_{\downarrow X}$ and Γ-$\mathbf{set}_{\downarrow Y}$ are equivalent, and in particular

$$\mathcal{B}({}_\Gamma Y) \cong \mathcal{B}({}_\Lambda X).$$

In particular if $X = Res_\Lambda^\Gamma Z$ for some finite Γ-set Z, then

$$\mathcal{B}\big({}_\Lambda(Res_\Lambda^\Gamma Z)\big) \cong \mathcal{B}\big({}_\Gamma(Ind_\Lambda^\Gamma Res_\Lambda^\Gamma Z)\big) \cong \mathcal{B}\Big({}_\Gamma\big((\Gamma/\Lambda) \times Z\big)\Big).$$

For $\Gamma = G \times G^{\mathrm{op}}$, $\Lambda = K \times H^{\mathrm{op}}$, and $Z = {}_G G_G$, this gives

$$\mathcal{B}({}_K G_H) \cong$$

From subgroups to G-sets

Moreover ${}_K G_H$ is the restriction to (K, H) of the (G, G)-biset G.

Lemma

Let Γ be a finite group, and Λ be a subgroup of Γ. Let X be a finite Λ-set, and let $Y = Ind_\Lambda^\Gamma X$. Then the categories Λ-$\mathbf{set}_{\downarrow X}$ and Γ-$\mathbf{set}_{\downarrow Y}$ are equivalent, and in particular

$$\mathcal{B}({}_\Gamma Y) \cong \mathcal{B}({}_\Lambda X).$$

In particular if $X = Res_\Lambda^\Gamma Z$ for some finite Γ-set Z, then

$$\mathcal{B}\big({}_\Lambda(Res_\Lambda^\Gamma Z)\big) \cong \mathcal{B}\big({}_\Gamma(Ind_\Lambda^\Gamma Res_\Lambda^\Gamma Z)\big) \cong \mathcal{B}\Big({}_\Gamma\big((\Gamma/\Lambda) \times Z\big)\Big).$$

For $\Gamma = G \times G^{\mathrm{op}}$, $\Lambda = K \times H^{\mathrm{op}}$, and $Z = {}_G G_G$, this gives

$$\mathcal{B}({}_K G_H) \cong \mathcal{B}\Big({}_G\big((G/K) \times G \times (G/H)\big)_G\Big).$$

Additive completion

Additive completion

More generally, one can then consider the following category Σ_G:

Additive completion

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.

Additive completion

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.
- if X and Y are finite G-sets, then $Y \times G \times X$ is a (G, G)-biset

Additive completion

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.
- if X and Y are finite G-sets, then $Y \times G \times X$ is a (G, G)-biset $(u(y, g, x)v = (uy, ugv, v^{-1}x)$, for $u, v, g \in G$, $y \in Y$ and $x \in X)$

Additive completion

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.
- if X and Y are finite G-sets, then $Y \times G \times X$ is a (G, G)-biset, and $Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big)$.

Additive completion

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.
- if X and Y are finite G-sets, then $Y \times G \times X$ is a (G, G)-biset, and $Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big)$.
- the composition in Σ_G is induced by

$$\left(\begin{array}{ccc} & S & \\ {\scriptstyle d}\swarrow & {\scriptstyle f}\downarrow & \searrow{\scriptstyle e} \\ Z & G & Y \end{array}\right) \circ \left(\begin{array}{ccc} & R & \\ {\scriptstyle a}\swarrow & {\scriptstyle c}\downarrow & \searrow{\scriptstyle b} \\ Y & G & X \end{array}\right) =$$

Additive completion

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.
- if X and Y are finite G-sets, then $Y \times G \times X$ is a (G, G)-biset, and $Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big)$.
- the composition in Σ_G is induced by

$$\left(\begin{tikzcd} & S \arrow[dl, "d"'] \arrow[d, "f"'] \arrow[dr, "e"] & \\ Z & G & Y \end{tikzcd}\right) \circ \left(\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[d, "c"'] \arrow[dr, "b"] & \\ Y & G & X \end{tikzcd}\right) = \left(\begin{tikzcd} & (S \times_{e,a} R)/G \arrow[dl, "u"'] \arrow[d, "w"'] \arrow[dr, "v"] & \\ Z & G & X \end{tikzcd}\right)$$

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.
- if X and Y are finite G-sets, then $Y \times G \times X$ is a (G, G)-biset, and $Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big)$.
- the composition in Σ_G is induced by

$$\left(\begin{tikzcd} & S \arrow[dl, "d"'] \arrow[d, "f"] \arrow[dr, "e"] & \\ Z & G & Y \end{tikzcd}\right) \circ \left(\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[d, "c"] \arrow[dr, "b"] & \\ Y & G & X \end{tikzcd}\right) = \left(\begin{tikzcd} & (S \times_{e,a} R)/G \arrow[dl, "u"'] \arrow[d, "w"] \arrow[dr, "v"] & \\ Z & G & X \end{tikzcd}\right)$$

where $(S \times_{e,a} R) = \{(s, r) \in S \times R \mid e(s) = a(r)\}$

Additive completion

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.
- if X and Y are finite G-sets, then $Y \times G \times X$ is a (G, G)-biset, and $Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big)$.
- the composition in Σ_G is induced by

$$\left(\begin{tikzcd} & S \arrow[dl, "d"'] \arrow[d, "f"] \arrow[dr, "e"] & \\ Z & G & Y \end{tikzcd}\right) \circ \left(\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[d, "c"] \arrow[dr, "b"] & \\ Y & G & X \end{tikzcd}\right) = \left(\begin{tikzcd} & (S \times_{e,a} R)/G \arrow[dl, "u"'] \arrow[d, "w"'] \arrow[dr, "v"] & \\ Z & G & X \end{tikzcd}\right)$$

where $(S \times_{e,a} R) = \{(s, r) \in S \times R \mid e(s) = a(r)\}$, with right G-action defined by $(s, r)g = (sg, g^{-1}r)$, for $g \in G$

Additive completion

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.
- if X and Y are finite G-sets, then $Y \times G \times X$ is a (G, G)-biset, and $Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big)$.
- the composition in Σ_G is induced by

$$\left(\begin{tikzcd} & S \arrow[dl, "d"'] \arrow[d, "f"] \arrow[dr, "e"] & \\ Z & G & Y \end{tikzcd}\right) \circ \left(\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[d, "c"] \arrow[dr, "b"] & \\ Y & G & X \end{tikzcd}\right) = \left(\begin{tikzcd} & (S \times_{e,a} R)/G \arrow[dl, "u"'] \arrow[d, "w"] \arrow[dr, "v"] & \\ Z & G & X \end{tikzcd}\right)$$

where $u\big((s,r)G\big) = d(s)$, $v\big((s,r)G\big) = b(r)$, $w\big((s,r)G\big) = f(s)c(r)$.

Additive completion

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.
- if X and Y are finite G-sets, then $Y \times G \times X$ is a (G,G)-biset, and $Hom_{\Sigma_G}(X,Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big)$.
- the composition in Σ_G is induced by

$$\left(\begin{tikzcd} & S \arrow[dl, "d"'] \arrow[d, "f"'] \arrow[dr, "e"] & \\ Z & G & Y \end{tikzcd}\right) \circ \left(\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[d, "c"'] \arrow[dr, "b"] & \\ Y & G & X \end{tikzcd}\right) = \left(\begin{tikzcd} & (S \times_{e,a} R)/G \arrow[dl, "u"'] \arrow[d, "w"'] \arrow[dr, "v"] & \\ Z & G & X \end{tikzcd}\right)$$

Additive completion

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.
- if X and Y are finite G-sets, then $Y \times G \times X$ is a (G, G)-biset, and $Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big)$.
- the composition in Σ_G is induced by

$$\left(\begin{tikzcd} & S \arrow[dl, "d"'] \arrow[d, "f"] \arrow[dr, "e"] & \\ Z & G & Y \end{tikzcd}\right) \circ \left(\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[d, "c"] \arrow[dr, "b"] & \\ Y & G & X \end{tikzcd}\right) = \left(\begin{tikzcd} & (S \times_{e,a} R)/G \arrow[dl, "u"'] \arrow[d, "w"] \arrow[dr, "v"] & \\ Z & G & X \end{tikzcd}\right)$$

- the identity morphism of the G-set X is $\left(\begin{tikzcd} & G \times X \arrow[dl, "i"'] \arrow[d, "k"] \arrow[dr, "j"] & \\ X & G & X \end{tikzcd}\right)$

Additive completion

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.
- if X and Y are finite G-sets, then $Y \times G \times X$ is a (G, G)-biset, and $Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big)$.
- the composition in Σ_G is induced by

$$\left(\begin{tikzcd} & S \arrow[dl, "d"'] \arrow[d, "f"'] \arrow[dr, "e"] & \\ Z & G & Y \end{tikzcd}\right) \circ \left(\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[d, "c"'] \arrow[dr, "b"] & \\ Y & G & X \end{tikzcd}\right) = \left(\begin{tikzcd} & (S \times_{e,a} R)/G \arrow[dl, "u"'] \arrow[d, "w"'] \arrow[dr, "v"] & \\ Z & G & X \end{tikzcd}\right)$$

- the identity morphism of the G-set X is $\left(\begin{tikzcd} & G \times X \arrow[dl, "i"'] \arrow[d, "k"'] \arrow[dr, "j"] & \\ X & G & X \end{tikzcd}\right)$,

  where $s(g, x)t = (sgt, t^{-1}x)$, for $s, t, g \in G$ and $x \in X$

Additive completion

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.
- if X and Y are finite G-sets, then $Y \times G \times X$ is a (G, G)-biset, and $Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big)$.
- the composition in Σ_G is induced by

$$\left(\begin{tikzcd} & S \arrow[dl, "d"'] \arrow[d, "f"] \arrow[dr, "e"] & \\ Z & G & Y \end{tikzcd}\right) \circ \left(\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[d, "c"] \arrow[dr, "b"] & \\ Y & G & X \end{tikzcd}\right) = \left(\begin{tikzcd} & (S \times_{e,a} R)/G \arrow[dl, "u"'] \arrow[d, "w"] \arrow[dr, "v"] & \\ Z & G & X \end{tikzcd}\right)$$

- the identity morphism of the G-set X is $\left(\begin{tikzcd} & G \times X \arrow[dl, "i"'] \arrow[d, "k"] \arrow[dr, "j"] & \\ X & G & X \end{tikzcd}\right)$,

  where $i(g, x) = gx$, $j(g, x) = x$, $k(g, x) = g$.

Additive completion

More generally, one can then consider the following category Σ_G:

- the objects of Σ_G are the finite G-sets.
- if X and Y are finite G-sets, then $Y \times G \times X$ is a (G, G)-biset, and $Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big)$.
- the composition in Σ_G is induced by

$$\left(\begin{tikzcd} & S \arrow[dl, "d"'] \arrow[d, "f"'] \arrow[dr, "e"] & \\ Z & G & Y \end{tikzcd}\right) \circ \left(\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[d, "c"'] \arrow[dr, "b"] & \\ Y & G & X \end{tikzcd}\right) = \left(\begin{tikzcd} & (S \times_{e,a} R)/G \arrow[dl, "u"'] \arrow[d, "w"'] \arrow[dr, "v"] & \\ Z & G & X \end{tikzcd}\right)$$

- the identity morphism of the G-set X is $\left(\begin{tikzcd} & G \times X \arrow[dl, "i"'] \arrow[d, "k"'] \arrow[dr, "j"] & \\ X & G & X \end{tikzcd}\right)$,

Additive completion

Additive completion

Recall that $\mathcal{C}_G$ is the category of subgroups of G, where $Hom_{\mathcal{C}_G}(H,K) = \mathcal{B}^G_{K,H}$.

Additive completion

Recall that $\mathcal{C}_G$ is the category of subgroups of G, where $Hom_{\mathcal{C}_G}(H,K)=\mathcal{B}^G_{K,H}$.
The additive completion of $\mathcal{C}_G$ is equivalent to the quotient $\underline{\Sigma}_G$ of Σ_G obtained by identifying morphisms $f,g\in Hom_{\Sigma_G}(X,Y)$

Additive completion

Recall that $\mathcal{C}_G$ is the category of subgroups of G, where $Hom_{\mathcal{C}_G}(H,K) = \mathcal{B}^G_{K,H}$.
The additive completion of $\mathcal{C}_G$ is equivalent to the quotient $\underline{\Sigma}_G$ of Σ_G obtained by identifying morphisms $f,g \in Hom_{\Sigma_G}(X,Y)$ such that $f-g$ is in the kernel of the map $\phi : \mathcal{B}\big({}_G(Y\times G\times X)_G\big) \to \mathcal{B}\big({}_G(Y\times X)_G\big)$

Additive completion

Recall that $\mathcal{C}_G$ is the category of subgroups of G, where $Hom_{\mathcal{C}_G}(H,K) = \mathcal{B}^G_{K,H}$.
The additive completion of $\mathcal{C}_G$ is equivalent to the quotient $\underline{\Sigma}_G$ of Σ_G obtained by identifying morphisms $f, g \in Hom_{\Sigma_G}(X,Y)$ such that $f - g$ is in the kernel of the map $\phi : \mathcal{B}\big({}_G(Y \times G \times X)_G\big) \to \mathcal{B}\big({}_G(Y \times X)_G\big)$ induced by the map

$$\left(\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[d, "c"'] \arrow[dr, "b"] & \\ Y & G & X \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[dr, "b"] & \\ Y & & X \end{tikzcd}\right)$$

Additive completion

Recall that $\mathcal{C}_G$ is the category of subgroups of G, where $Hom_{\mathcal{C}_G}(H,K) = \mathcal{B}^G_{K,H}$.
The additive completion of $\mathcal{C}_G$ is equivalent to the quotient $\underline{\Sigma}_G$ of Σ_G obtained by identifying morphisms $f, g \in Hom_{\Sigma_G}(X,Y)$ such that $f-g$ is in the kernel of the map $\phi : \mathcal{B}\big({}_G(Y \times G \times X)_G\big) \to \mathcal{B}\big({}_G(Y \times X)_G\big)$ induced by the map

$$\left(\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[d, "c"] \arrow[dr, "b"] & \\ Y & G & X \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[dr, "b"] & \\ Y & & X \end{tikzcd}\right)$$

It follows that the category $\mathcal{F}_G$ is equivalent to the category of additive functors $\underline{\Sigma}_G \to \mathbb{Z}$-**Mod**.

Σ_G revisited

Σ_G revisited

Recall that the objects of Σ_G are the finite G-sets

Σ_G revisited

Recall that the objects of Σ_G are the finite G-sets, and that

$$Hom_{\Sigma_G}(X,Y)=\mathcal{B}\big({}_G(Y\times G\times X)_G\big).$$

Σ_G revisited

Recall that the objects of Σ_G are the finite G-sets, and that

$$Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big).$$

Now $G = Ind_{\Delta(G)}^{G \times G^{\mathrm{op}}} \bullet$

Σ_G revisited

Recall that the objects of Σ_G are the finite G-sets, and that

$$Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big).$$

Now $G = Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}\bullet$, where $\Delta(G) = \{(g, g^{-1}) \mid g \in G\}$

Σ_G revisited

Recall that the objects of Σ_G are the finite G-sets, and that

$$Hom_{\Sigma_G}(X,Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big).$$

Now $G = Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}\bullet$, thus $Y \times G \times X \cong Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}(Y \times X)$.

Σ_G revisited

Recall that the objects of Σ_G are the finite G-sets, and that

$$Hom_{\Sigma_G}(X,Y) = \mathcal{B}\big({}_G(Y\times G\times X)_G\big).$$

Now $G = Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}\bullet$, thus $Y\times G\times X \cong Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}(Y\times X)$.
Hence $\mathcal{B}\big({}_G(Y\times G\times X)_G\big) \cong \mathcal{B}\big({}_G(Y\times X)\big)$.

Σ_G revisited

Recall that the objects of Σ_G are the finite G-sets, and that

$$Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big).$$

Now $G = Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}\bullet$, thus $Y \times G \times X \cong Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}(Y \times X)$.
Hence $\mathcal{B}\big({}_G(Y \times G \times X)_G\big) \cong \mathcal{B}\big({}_G(Y \times X)\big)$.
Keeping track of the composition in Σ_G

Σ_G revisited

Recall that the objects of Σ_G are the finite G-sets, and that

$$Hom_{\Sigma_G}(X,Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big).$$

Now $G = Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}\bullet$, thus $Y \times G \times X \cong Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}(Y \times X)$.
Hence $\mathcal{B}\big({}_G(Y \times G \times X)_G\big) \cong \mathcal{B}\big({}_G(Y \times X)\big)$.
Keeping track of the composition in Σ_G shows that Σ_G is equivalent to the category $\mathcal{S}_G$ of Lindner

Σ_G revisited

Recall that the objects of Σ_G are the finite G-sets, and that

$$Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big).$$

Now $G = Ind_{\Delta(G)}^{G \times G^{\mathrm{op}}} \bullet$, thus $Y \times G \times X \cong Ind_{\Delta(G)}^{G \times G^{\mathrm{op}}}(Y \times X)$.
Hence $\mathcal{B}\big({}_G(Y \times G \times X)_G\big) \cong \mathcal{B}\big({}_G(Y \times X)\big)$.
Keeping track of the composition in Σ_G shows that Σ_G is equivalent to the category $\mathcal{S}_G$ of Lindner, hence the additive functors from Σ_G to $\mathbb{Z}$-**Mod** are the Mackey functors for G.

Σ_G revisited

Recall that the objects of Σ_G are the finite G-sets, and that

$$Hom_{\Sigma_G}(X, Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big).$$

Now $G = Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}} \bullet$, thus $Y \times G \times X \cong Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}(Y \times X)$.
Hence $\mathcal{B}\big({}_G(Y \times G \times X)_G\big) \cong \mathcal{B}\big({}_G(Y \times X)\big)$.
Keeping track of the composition in Σ_G shows that Σ_G is equivalent to the category $\mathcal{S}_G$ of Lindner, hence the additive functors from Σ_G to $\mathbb{Z}$-**Mod** are the Mackey functors for G.
To describe Σ_G in those terms

Σ_G revisited

Recall that the objects of Σ_G are the finite G-sets, and that

$$Hom_{\Sigma_G}(X,Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big).$$

Now $G = Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}\bullet$, thus $Y \times G \times X \cong Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}(Y \times X)$.
Hence $\mathcal{B}\big({}_G(Y \times G \times X)_G\big) \cong \mathcal{B}\big({}_G(Y \times X)\big)$.
Keeping track of the composition in Σ_G shows that Σ_G is equivalent to the category $\mathcal{S}_G$ of Lindner, hence the additive functors from Σ_G to $\mathbb{Z}$-**Mod** are the Mackey functors for G.
To describe $\underline{\Sigma}_G$ in those terms, it remains to see which morphisms in $\mathcal{S}_G$ vanish in $\underline{\Sigma}_G$.

Σ_G revisited

Recall that the objects of Σ_G are the finite G-sets, and that

$$Hom_{\Sigma_G}(X,Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big).$$

Now $G = Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}\bullet$, thus $Y \times G \times X \cong Ind_{\Delta(G)}^{G\times G^{\mathrm{op}}}(Y \times X)$.
Hence $\mathcal{B}\big({}_G(Y \times G \times X)_G\big) \cong \mathcal{B}\big({}_G(Y \times X)\big)$.
Keeping track of the composition in Σ_G shows that Σ_G is equivalent to the category $\mathcal{S}_G$ of Lindner, hence the additive functors from Σ_G to $\mathbb{Z}$-**Mod** are the Mackey functors for G.
To describe $\underline{\Sigma}_G$ in those terms, it remains to see which morphisms in $\mathcal{S}_G$ vanish in $\underline{\Sigma}_G$.

$\underline{\Sigma}_G$ revisited

The morphism $f = \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ Y & & X\end{tikzcd}\right) - \left(\begin{tikzcd}[column sep=small] & Z' \arrow[dl, "b'"'] \arrow[dr, "a'"] & \\ Y & & X\end{tikzcd}\right)$ of $\mathcal{S}_G$ is mapped

to the morphism

Σ_G revisited

The morphism $f = \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}\right) - \left(\begin{tikzcd}[column sep=small] & Z' \arrow[dl, "b'"'] \arrow[dr, "a'"] & \\ Y & & X \end{tikzcd}\right)$ of $\mathcal{S}_G$ is mapped

to the morphism $\left(\begin{tikzcd}[column sep=small] & G \times Z \arrow[dl, "\beta"'] \arrow[d, "\gamma"'] \arrow[dr, "\alpha"] & \\ Y & G & X \end{tikzcd}\right) - \left(\begin{tikzcd}[column sep=small] & G \times Z' \arrow[dl, "\beta'"'] \arrow[d, "\gamma'"'] \arrow[dr, "\alpha'"] & \\ Y & G & X \end{tikzcd}\right)$ of Σ_G

$\underline{\Sigma}_G$ revisited

The morphism $f = \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}\right) - \left(\begin{tikzcd}[column sep=small] & Z' \arrow[dl, "b'"'] \arrow[dr, "a'"] & \\ Y & & X \end{tikzcd}\right)$ of $\mathcal{S}_G$ is mapped to the morphism $\left(\begin{tikzcd}[column sep=small] & G \times Z \arrow[dl, "\beta"'] \arrow[d, "\gamma"'] \arrow[dr, "\alpha"] & \\ Y & G & X \end{tikzcd}\right) - \left(\begin{tikzcd}[column sep=small] & G \times Z' \arrow[dl, "\beta'"'] \arrow[d, "\gamma'"'] \arrow[dr, "\alpha'"] & \\ Y & G & X \end{tikzcd}\right)$ of Σ_G, where the (G, G)-action on $G \times Z$ is given by $s(g, z)t = (sgt, t^{-1}z)$ for $s, t, g \in G$ and $z \in Z$

Σ_G revisited

The morphism $f = \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}\right) - \left(\begin{tikzcd}[column sep=small] & Z' \arrow[dl, "b'"'] \arrow[dr, "a'"] & \\ Y & & X \end{tikzcd}\right)$ of $\mathcal{S}_G$ is mapped to the morphism $\left(\begin{tikzcd}[column sep=small] & G \times Z \arrow[dl, "\beta"'] \arrow[d, "\gamma"'] \arrow[dr, "\alpha"] & \\ Y & G & X \end{tikzcd}\right) - \left(\begin{tikzcd}[column sep=small] & G \times Z' \arrow[dl, "\beta'"'] \arrow[d, "\gamma'"'] \arrow[dr, "\alpha'"] & \\ Y & G & X \end{tikzcd}\right)$ of Σ_G, where the (G, G)-action on $G \times Z$ is given by $s(g, z)t = (sgt, t^{-1}z)$ for $s, t, g \in G$ and $z \in Z$, and where $(\beta, \gamma, \alpha)(g, z) = \big(gb(z), g, a(z)\big)$.

$\underline{\Sigma}_G$ revisited

The morphism $f = \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}\right) - \left(\begin{tikzcd}[column sep=small] & Z' \arrow[dl, "b'"'] \arrow[dr, "a'"] & \\ Y & & X \end{tikzcd}\right)$ of $\mathcal{S}_G$ is mapped

to the morphism $\left(\begin{tikzcd}[column sep=small] & G \times Z \arrow[dl, "\beta"'] \arrow[d, "\gamma"'] \arrow[dr, "\alpha"] & \\ Y & G & X \end{tikzcd}\right) - \left(\begin{tikzcd}[column sep=small] & G \times Z' \arrow[dl, "\beta'"'] \arrow[d, "\gamma'"'] \arrow[dr, "\alpha'"] & \\ Y & G & X \end{tikzcd}\right)$ of Σ_G,

This gives 0 in $\underline{\Sigma}_G$ if and only if there is an isomorphism

$$\theta : \left(\begin{tikzcd}[column sep=small] & G \times Z \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ Y & & X \end{tikzcd}\right) \to \left(\begin{tikzcd}[column sep=small] & G \times Z' \arrow[dl, "\beta'"'] \arrow[dr, "\alpha'"] & \\ Y & & X \end{tikzcd}\right)$$

of (G, G)-bisets over $Y \times X$.

Σ_G revisited

Σ_G revisited

Equivalently f is equal to

$$(*) \qquad \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd} \right) - \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "u*b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd} \right)$$

Σ_G revisited

Equivalently f is equal to

$$(*)\qquad \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}\right) - \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "u*b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}\right)$$

where $u : Z \to G^c$ is a morphism of G-sets

Σ_G revisited

Equivalently f is equal to

$$(*) \qquad \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd} \right) - \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "u*b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd} \right)$$

where $u : Z \to G^c$ is a morphism of G-sets, where $G^c = G$ acted on by conjugation

Σ_G revisited

Equivalently f is equal to

$$(*) \qquad \begin{pmatrix} & Z & \\ {}^{b}\swarrow & & \searrow^{a} \\ Y & & X \end{pmatrix} - \begin{pmatrix} & Z & \\ {}^{u*b}\swarrow & & \searrow^{a} \\ Y & & X \end{pmatrix}$$

where $u : Z \to G^c$ is a morphism of G-sets, where $G^c = G$ acted on by conjugation, and $(u * b)(z) = u(z)b(z)$ for $z \in Z$.

Σ_G revisited

Equivalently f is equal to

$$(*) \qquad \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}\right) - \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "u*b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}\right)$$

where $u : Z \to G^c$ is a morphism of G-sets, where $G^c = G$ acted on by conjugation, and $(u * b)(z) = u(z)b(z)$ for $z \in Z$.

Theorem

Equivalently f is equal to

$$(*) \qquad \left(\begin{array}{ccc} & Z & \\ {\scriptstyle b}\swarrow & & \searrow{\scriptstyle a} \\ Y & & X \end{array} \right) - \left(\begin{array}{ccc} & Z & \\ {\scriptstyle u*b}\swarrow & & \searrow{\scriptstyle a} \\ Y & & X \end{array} \right)$$

where $u : Z \to G^c$ is a morphism of G-sets, where $G^c = G$ acted on by conjugation, and $(u * b)(z) = u(z)b(z)$ for $z \in Z$.

Theorem

Let $\underline{\mathcal{S}}_G$ be the quotient category of $\mathcal{S}_G$

Σ_G revisited

Equivalently f is equal to

$$(*)\qquad \left(\begin{array}{ccc} & Z & \\ {}^{b}\swarrow & & \searrow^{a} \\ Y & & X\end{array}\right) - \left(\begin{array}{ccc} & Z & \\ {}^{u*b}\swarrow & & \searrow^{a} \\ Y & & X\end{array}\right)$$

where $u : Z \to G^c$ is a morphism of G-sets, where $G^c = G$ acted on by conjugation, and $(u * b)(z) = u(z)b(z)$ for $z \in Z$.

Theorem

Let $\underline{\mathcal{S}}_G$ be the quotient category of $\mathcal{S}_G$, defined for finite G-sets X and Y by

$$Hom_{\underline{\mathcal{S}}_G}(X, Y) = \mathcal{B}\big({}_G(Y \times X)\big)/K(Y, X),$$

$\underline{\Sigma}_G$ revisited

Equivalently f is equal to

$$(*) \qquad \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}\right) - \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "u*b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}\right)$$

where $u : Z \to G^c$ is a morphism of G-sets, where $G^c = G$ acted on by conjugation, and $(u * b)(z) = u(z)b(z)$ for $z \in Z$.

Theorem

Let $\underline{\mathcal{S}}_G$ be the quotient category of $\mathcal{S}_G$, defined for finite G-sets X and Y by

$$Hom_{\underline{\mathcal{S}}_G}(X, Y) = \mathcal{B}\big({}_G(Y \times X)\big)/K(Y, X),$$

Σ_G revisited

Equivalently f is equal to

$$(*) \qquad \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}\right) - \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "u*b"'] \arrow[dr, "a"] & \\ Y & & X \end{tikzcd}\right)$$

where $u : Z \to G^c$ is a morphism of G-sets, where $G^c = G$ acted on by conjugation, and $(u * b)(z) = u(z)b(z)$ for $z \in Z$.

Theorem

Let $\underline{\mathcal{S}}_G$ be the quotient category of $\mathcal{S}_G$, defined for finite G-sets X and Y by

$$Hom_{\underline{\mathcal{S}}_G}(X, Y) = \mathcal{B}\big({}_G(Y \times X)\big)/K(Y, X),$$

where $K(Y, X)$ is the subgroup generated by differences (*).

$\underline{\Sigma}_G$ revisited

Equivalently f is equal to

$$(*) \qquad \left(\begin{array}{ccc} & Z & \\ {}^{b}\swarrow & & \searrow^{a} \\ Y & & X \end{array} \right) - \left(\begin{array}{ccc} & Z & \\ {}^{u*b}\swarrow & & \searrow^{a} \\ Y & & X \end{array} \right)$$

where $u : Z \to G^c$ is a morphism of G-sets, where $G^c = G$ acted on by conjugation, and $(u * b)(z) = u(z)b(z)$ for $z \in Z$.

Theorem

Let $\underline{\mathcal{S}}_G$ be the quotient category of $\mathcal{S}_G$, defined for finite G-sets X and Y by

$$Hom_{\underline{\mathcal{S}}_G}(X, Y) = \mathcal{B}\big({}_G(Y \times X)\big)/K(Y, X),$$

where $K(Y, X)$ is the subgroup generated by differences (*).
*Then $\mathcal{F}_G$ is equivalent to the category of additive functors $\underline{\mathcal{S}}_G \to \mathbb{Z}$-**Mod**.*

Fused G-sets

Fused G-sets

Let Z be a G-set.

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\text{-}\mathbf{set}}(Z, G^c)$.

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\textbf{-set}}(Z, G^c)$. Then $\Gamma_G(Z)$ is a group by pointwise multiplication $(u \cdot v)(z) = u(z)v(z)$, for $u, v \in \Gamma_G(Z)$ and $z \in Z$

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\text{-}\mathbf{set}}(Z, G^c)$. Then $\Gamma_G(Z)$ is a group by pointwise multiplication, e.g. $\Gamma_G(G/H) \cong C_G(H)$ for $H \leq G$

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\textbf{-set}}(Z, G^c)$. Then $\Gamma_G(Z)$ is a group by pointwise multiplication, and for any G-set Y, this group acts on $Hom_{G\textbf{-set}}(Z, Y)$

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\textbf{-set}}(Z, G^c)$. Then $\Gamma_G(Z)$ is a group by pointwise multiplication, and for any G-set Y, this group acts on $Hom_{G\textbf{-set}}(Z, Y)$ by
$(u * f)(z) = u(z)f(z)$ for $u \in \Gamma_G(Z)$, $f \in Hom_{G\textbf{-set}}(Z, Y)$, and $z \in Z$.

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\textbf{-set}}(Z, G^c)$. Then $\Gamma_G(Z)$ is a group by pointwise multiplication, and for any G-set Y, this group acts on $Hom_{G\textbf{-set}}(Z, Y)$ by
$(u * f)(z) = u(z)f(z)$ for $u \in \Gamma_G(Z)$, $f \in Hom_{G\textbf{-set}}(Z, Y)$, and $z \in Z$.

Lemma

Let X, Y, Z be finite G-sets.

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\text{-}\mathbf{set}}(Z, G^c)$. Then $\Gamma_G(Z)$ is a group by pointwise multiplication, and for any G-set Y, this group acts on $Hom_{G\text{-}\mathbf{set}}(Z, Y)$ by
$(u * f)(z) = u(z)f(z)$ for $u \in \Gamma_G(Z)$, $f \in Hom_{G\text{-}\mathbf{set}}(Z, Y)$, and $z \in Z$.

Lemma

Let X, Y, Z be finite G-sets. Let $u : Z \to G^c$

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\textbf{-set}}(Z, G^c)$. Then $\Gamma_G(Z)$ is a group by pointwise multiplication, and for any G-set Y, this group acts on $Hom_{G\textbf{-set}}(Z, Y)$ by
$(u * f)(z) = u(z)f(z)$ for $u \in \Gamma_G(Z)$, $f \in Hom_{G\textbf{-set}}(Z, Y)$, and $z \in Z$.

Lemma

Let X, Y, Z be finite G-sets. Let $u : Z \to G^c$, $v : Y \to G^c$

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\text{-}\mathbf{set}}(Z, G^c)$. Then $\Gamma_G(Z)$ is a group by pointwise multiplication, and for any G-set Y, this group acts on $Hom_{G\text{-}\mathbf{set}}(Z, Y)$ by
$(u * f)(z) = u(z)f(z)$ for $u \in \Gamma_G(Z)$, $f \in Hom_{G\text{-}\mathbf{set}}(Z, Y)$, and $z \in Z$.

Lemma

Let X, Y, Z be finite G-sets. Let $u : Z \to G^c$, $v : Y \to G^c$, $f : Z \to Y$

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\text{-}\mathbf{set}}(Z, G^c)$. Then $\Gamma_G(Z)$ is a group by pointwise multiplication, and for any G-set Y, this group acts on $Hom_{G\text{-}\mathbf{set}}(Z, Y)$ by
$(u * f)(z) = u(z)f(z)$ for $u \in \Gamma_G(Z)$, $f \in Hom_{G\text{-}\mathbf{set}}(Z, Y)$, and $z \in Z$.

Lemma

Let X, Y, Z be finite G-sets. Let $u : Z \to G^c$, $v : Y \to G^c$, $f : Z \to Y$ and $g : Y \to X$ be morphisms of G-sets.

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\text{-}\mathbf{set}}(Z, G^c)$. Then $\Gamma_G(Z)$ is a group by pointwise multiplication, and for any G-set Y, this group acts on $Hom_{G\text{-}\mathbf{set}}(Z, Y)$ by
$(u * f)(z) = u(z)f(z)$ for $u \in \Gamma_G(Z)$, $f \in Hom_{G\text{-}\mathbf{set}}(Z, Y)$, and $z \in Z$.

Lemma

Let X, Y, Z be finite G-sets. Let $u : Z \to G^c$, $v : Y \to G^c$, $f : Z \to Y$ and $g : Y \to X$ be morphisms of G-sets. Then

$$(v * g) \circ (u * f) =$$

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\text{-}\mathbf{set}}(Z, G^c)$. Then $\Gamma_G(Z)$ is a group by pointwise multiplication, and for any G-set Y, this group acts on $Hom_{G\text{-}\mathbf{set}}(Z, Y)$ by
$(u * f)(z) = u(z)f(z)$ for $u \in \Gamma_G(Z)$, $f \in Hom_{G\text{-}\mathbf{set}}(Z, Y)$, and $z \in Z$.

Lemma

Let X, Y, Z be finite G-sets. Let $u : Z \to G^c$, $v : Y \to G^c$, $f : Z \to Y$ and $g : Y \to X$ be morphisms of G-sets. Then

$$(v * g) \circ (u * f) = \big(u \cdot (v \circ f)\big) * (g \circ f).$$

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\text{-}\mathbf{set}}(Z, G^c)$. Then $\Gamma_G(Z)$ is a group by pointwise multiplication, and for any G-set Y, this group acts on $Hom_{G\text{-}\mathbf{set}}(Z, Y)$ by
$(u * f)(z) = u(z)f(z)$ for $u \in \Gamma_G(Z)$, $f \in Hom_{G\text{-}\mathbf{set}}(Z, Y)$, and $z \in Z$.

Lemma

Let X, Y, Z be finite G-sets. Let $u : Z \to G^c$, $v : Y \to G^c$, $f : Z \to Y$ and $g : Y \to X$ be morphisms of G-sets. Then

$$(v * g) \circ (u * f) = \big(u \cdot (v \circ f)\big) * (g \circ f).$$

Definition

The category G-$\underline{\mathbf{set}}$ of fused G-sets

Fused G-sets

Let Z be a G-set. Set $\Gamma_G(Z) = Hom_{G\text{-}\mathbf{set}}(Z, G^c)$. Then $\Gamma_G(Z)$ is a group by pointwise multiplication, and for any G-set Y, this group acts on $Hom_{G\text{-}\mathbf{set}}(Z, Y)$ by
$(u * f)(z) = u(z)f(z)$ for $u \in \Gamma_G(Z)$, $f \in Hom_{G\text{-}\mathbf{set}}(Z, Y)$, and $z \in Z$.

Lemma

Let X, Y, Z be finite G-sets. Let $u : Z \to G^c$, $v : Y \to G^c$, $f : Z \to Y$ and $g : Y \to X$ be morphisms of G-sets. Then

$$(v * g) \circ (u * f) = \big(u \cdot (v \circ f)\big) * (g \circ f).$$

Definition

The category G-$\underline{\mathbf{set}}$ of fused G-sets is the quotient category of G-$\mathbf{set}$ defined for finite G-sets X and Y by

$$Hom_{G\text{-}\underline{\mathbf{set}}}(X, Y) = \Gamma_G(X) \backslash Hom_{G\text{-}\mathbf{set}}(X, Y).$$

Remarks on fused G-sets

Remarks on fused G-sets

- For a G-set Y, set $Y^I = Y \times G^c$.

Remarks on fused G-sets

- For a G-set Y, set $Y^l = Y \times G^c$. Then there is a morphism $p : Y^l \to Y \times Y$ defined by $p(y, g) = (y, gy)$, for $y \in Y$ and $g \in G$

Remarks on fused G-sets

- For a G-set Y, set $Y^I = Y \times G^c$. Then there is a morphism $p : Y^I \to Y \times Y$ defined by $p(y, g) = (y, gy)$, for $y \in Y$ and $g \in G$, and a morphism $i : Y \to Y^I$ defined by $i(y) = (y, 1)$.

Remarks on fused G-sets

- For a G-set Y, set $Y^I = Y \times G^c$. Then there is a morphism $p : Y^I \to Y \times Y$ defined by $p(y,g) = (y, gy)$, for $y \in Y$ and $g \in G$, and a morphism $i : Y \to Y^I$ defined by $i(y) = (y, 1)$.
  The composition $p \circ i$ is the diagonal map $Y \to Y \times Y$.

Remarks on fused G-sets

- For a G-set Y, set $Y^I = Y \times G^c$. Then there is a morphism $p : Y^I \to Y \times Y$ defined by $p(y, g) = (y, gy)$, for $y \in Y$ and $g \in G$, and a morphism $i : Y \to Y^I$ defined by $i(y) = (y, 1)$.
  The composition $p \circ i$ is the diagonal map $Y \to Y \times Y$.
  Two morphisms $f, g : X \to Y$ in G-**set** are identified in G-<u>**set**</u>

Remarks on fused G-sets

- For a G-set Y, set $Y^I = Y \times G^c$. Then there is a morphism $p : Y^I \to Y \times Y$ defined by $p(y, g) = (y, gy)$, for $y \in Y$ and $g \in G$, and a morphism $i : Y \to Y^I$ defined by $i(y) = (y, 1)$.
  The composition $p \circ i$ is the diagonal map $Y \to Y \times Y$.
  Two morphisms $f, g : X \to Y$ in G-**set** are identified in G-<u>**set**</u> if and only if there exists $\varphi : X \to Y^I$ such that the diagram

$$\begin{tikzcd} & Y^I \arrow[d, "p"] \\ X \arrow[ur, "\varphi"] \arrow[r, "(f,g)"'] & Y \times Y \end{tikzcd}$$

  is commutative.

Remarks on fused G-sets

- For a G-set Y, set $Y^I = Y \times G^c$. Then there is a morphism $p : Y^I \to Y \times Y$ defined by $p(y,g) = (y,gy)$, for $y \in Y$ and $g \in G$, and a morphism $i : Y \to Y^I$ defined by $i(y) = (y,1)$.
  The composition $p \circ i$ is the diagonal map $Y \to Y \times Y$.
  Two morphisms $f,g : X \to Y$ in G-**set** are identified in G-__**set**__ if and only if there exists $\varphi : X \to Y^I$ such that the diagram

$$\begin{tikzcd} & Y^I \arrow[d, "p"] \\ X \arrow[ur, "\varphi"] \arrow[r, "{(f,g)}"'] & Y \times Y \end{tikzcd}$$

  is commutative.
- Disjoint union of G-sets is a coproduct in G-__**set**__

- For a G-set Y, set $Y^l = Y \times G^c$. Then there is a morphism $p : Y^l \to Y \times Y$ defined by $p(y,g) = (y, gy)$, for $y \in Y$ and $g \in G$, and a morphism $i : Y \to Y^l$ defined by $i(y) = (y,1)$.
  The composition $p \circ i$ is the diagonal map $Y \to Y \times Y$.
  Two morphisms $f, g : X \to Y$ in G-**set** are identified in G-<u>**set**</u> if and only if there exists $\varphi : X \to Y^l$ such that the diagram

$$\begin{tikzcd} & Y^l \arrow[d, "p"] \\ X \arrow[ur, "\varphi"] \arrow[r, "(f,g)"'] & Y \times Y \end{tikzcd}$$

  is commutative.
- Disjoint union of G-sets is a coproduct in G-<u>**set**</u>: indeed for G-sets X, Y, and Z

$$Hom_{G\text{-}\underline{\mathbf{set}}}(X \sqcup Y, Z) \quad \cong \quad \Gamma_G(X \sqcup Y) \backslash Hom_{G\text{-}\mathbf{set}}(X \sqcup Y, Z)$$

Remarks on fused G-sets

- For a G-set Y, set $Y^I = Y \times G^c$. Then there is a morphism $p : Y^I \to Y \times Y$ defined by $p(y,g) = (y, gy)$, for $y \in Y$ and $g \in G$, and a morphism $i : Y \to Y^I$ defined by $i(y) = (y, 1)$.
  The composition $p \circ i$ is the diagonal map $Y \to Y \times Y$.
  Two morphisms $f, g : X \to Y$ in G-**set** are identified in G-<u>**set**</u> if and only if there exists $\varphi : X \to Y^I$ such that the diagram

$$\begin{tikzcd} & Y^I \arrow[d, "p"] \\ X \arrow[ur, "\varphi"] \arrow[r, "(f,g)"'] & Y \times Y \end{tikzcd}$$

  is commutative.
- Disjoint union of G-sets is a coproduct in G-<u>**set**</u>: indeed for G-sets X, Y, and Z

$$\begin{aligned} Hom_{G\text{-}\underline{\mathbf{set}}}(X \sqcup Y, Z) &\cong \Gamma_G(X \sqcup Y) \backslash Hom_{G\text{-}\mathbf{set}}(X \sqcup Y, Z) \\ &\cong \big(\Gamma_G(X) \times \Gamma_G(Y)\big) \setminus \big(Hom_{G\text{-}\mathbf{set}}(X, Z) \times Hom_{G\text{-}\mathbf{set}}(Y, Z)\big) \end{aligned}$$

Remarks on fused G-sets

- For a G-set Y, set $Y^I = Y \times G^c$. Then there is a morphism $p : Y^I \to Y \times Y$ defined by $p(y, g) = (y, gy)$, for $y \in Y$ and $g \in G$, and a morphism $i : Y \to Y^I$ defined by $i(y) = (y, 1)$.
  The composition $p \circ i$ is the diagonal map $Y \to Y \times Y$.
  Two morphisms $f, g : X \to Y$ in G-**set** are identified in G-**<u>set</u>** if and only if there exists $\varphi : X \to Y^I$ such that the diagram
  $$\begin{tikzcd} & Y^I \arrow[d, "p"] \\ X \arrow[ur, "\varphi"] \arrow[r, "(f,g)"'] & Y \times Y \end{tikzcd}$$
  is commutative.
- Disjoint union of G-sets is a coproduct in G-**<u>set</u>**: indeed for G-sets X, Y, and Z

$$\begin{aligned} Hom_{G\text{-}\underline{\mathbf{set}}}(X \sqcup Y, Z) &\cong \Gamma_G(X \sqcup Y)\backslash Hom_{G\text{-}\mathbf{set}}(X \sqcup Y, Z) \\ &\cong \big(\Gamma_G(X) \times \Gamma_G(Y)\big) \setminus \big(Hom_{G\text{-}\mathbf{set}}(X, Z) \times Hom_{G\text{-}\mathbf{set}}(Y, Z)\big) \\ &\cong Hom_{G\text{-}\underline{\mathbf{set}}}(X, Z) \times Hom_{G\text{-}\underline{\mathbf{set}}}(Y, Z). \end{aligned}$$

Weak pullback of fused G-sets

Weak pullback of fused G-sets

Let $a \mapsto \underline{a}$ denote the quotient functor from G-**set** to G-$\underline{\textbf{set}}$.

Weak pullback of fused G-sets

Let $a \mapsto \underline{a}$ denote the quotient functor from G-**set** to G-$\underline{\textbf{set}}$.

Lemma

Let $\underline{a} : X \to Z$ *and* $\underline{b} : Y \to Z$ *be morphisms in* G-$\underline{\textbf{set}}$.

Weak pullback of fused G-sets

Let $a \mapsto \underline{a}$ denote the quotient functor from G-**set** to G-$\underline{\textbf{set}}$.

Lemma

Let $\underline{a} : X \to Z$ and $\underline{b} : Y \to Z$ be morphisms in G-$\underline{\textbf{set}}$. Then the pullback $X \times_{a,b} Y = \{(x, y) \in X \times Y \mid a(x) = b(y)\}$ only depends on $\underline{a}$ and $\underline{b}$, up to isomorphism in G-$\underline{\textbf{set}}$.

Weak pullback of fused G-sets

Let $a \mapsto \underline{a}$ denote the quotient functor from G-**set** to G-$\underline{\textbf{set}}$.

Lemma

Let $\underline{a} : X \to Z$ and $\underline{b} : Y \to Z$ be morphisms in G-$\underline{\textbf{set}}$. Then the pullback $X \times_{a,b} Y = \{(x, y) \in X \times Y \mid a(x) = b(y)\}$ only depends on $\underline{a}$ and $\underline{b}$, up to isomorphism in G-$\underline{\textbf{set}}$. It will be denoted by $X \times_{\underline{a},\underline{b}} Y$.

Weak pullback of fused G-sets

Let $a \mapsto \underline{a}$ denote the quotient functor from G-**set** to G-$\underline{\textbf{set}}$.

Lemma

Let $\underline{a} : X \to Z$ and $\underline{b} : Y \to Z$ be morphisms in G-$\underline{\textbf{set}}$. Then the pullback $X \times_{a,b} Y = \{(x, y) \in X \times Y \mid a(x) = b(y)\}$ only depends on $\underline{a}$ and $\underline{b}$, up to isomorphism in G-$\underline{\textbf{set}}$. It will be denoted by $X \times_{\underline{a},\underline{b}} Y$.

Proposition

1. *The square*

$$\begin{tikzcd}
 & X \times_{\underline{a},\underline{b}} Y \arrow[dl, "\underline{p}"'] \arrow[dr, "\underline{q}"] & \\
X \arrow[dr, "\underline{a}"'] & & Y \arrow[dl, "\underline{b}"] \\
 & Z &
\end{tikzcd}$$

is a weak pullback in G-$\underline{\textbf{set}}$.

Weak pullback of fused G-sets

Let $a \mapsto \underline{a}$ denote the quotient functor from G-**set** to G-$\underline{\textbf{set}}$.

Lemma

Let $\underline{a} : X \to Z$ and $\underline{b} : Y \to Z$ be morphisms in G-$\underline{\textbf{set}}$. Then the pullback $X \times_{a,b} Y = \{(x,y) \in X \times Y \mid a(x) = b(y)\}$ only depends on $\underline{a}$ and $\underline{b}$, up to isomorphism in G-$\underline{\textbf{set}}$. It will be denoted by $X \times_{\underline{a},\underline{b}} Y$.

Proposition

1. *The square*

$$\begin{tikzcd}
 & X \times_{\underline{a},\underline{b}} Y \arrow[dl, "\underline{p}"'] \arrow[dr, "\underline{q}"] & \\
X \arrow[dr, "\underline{a}"'] & & Y \arrow[dl, "\underline{b}"] \\
 & Z &
\end{tikzcd}$$

   is a weak pullback in G-$\underline{\textbf{set}}$.

2. *The category $\underline{\mathcal{S}}_G$ is the category of spans of fused G-sets.*

Fused Mackey functors (*à la Dress*)

Fused Mackey functors

Definition

A fused Mackey functor is a bivariant functor G-$\underline{\mathbf{set}} \to \mathbb{Z}$-$\mathbf{Mod}$ such that:

Fused Mackey functors

Definition

A fused Mackey functor is a bivariant functor G-$\underline{\mathbf{set}} \to \mathbb{Z}$-**Mod** such that:

1. for any finite G-sets X and Y, the maps

$$M(X \sqcup Y) \underset{(M_*(\underline{i}_X),M_*(\underline{i}_Y))}{\overset{(M^*(\underline{i}_X),M^*(\underline{i}_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$

Fused Mackey functors

Definition

A fused Mackey functor is a bivariant functor G-**set**$\to \mathbb{Z}$-**Mod** such that:

1. for any finite G-sets X and Y, the maps

$$M(X \sqcup Y) \underset{(M_*(\underline{i}_X),M_*(\underline{i}_Y))}{\overset{(M^*(\underline{i}_X),M^*(\underline{i}_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$

   are inverse to each other

Fused Mackey functors

Definition

A fused Mackey functor is a bivariant functor G-**set**$\to \mathbb{Z}$-**Mod** such that:

1. for any finite G-sets X and Y, the maps
$$M(X \sqcup Y) \underset{(M_*(\underline{i}_X), M_*(\underline{i}_Y))}{\overset{(M^*(\underline{i}_X), M^*(\underline{i}_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$
are inverse to each other, where $X \overset{i_X}{\hookrightarrow} X \sqcup Y \overset{i_Y}{\hookleftarrow} Y$.

Fused Mackey functors

Definition

A fused Mackey functor is a bivariant functor G-**set**$\to \mathbb{Z}$-**Mod** such that:

1. for any finite G-sets X and Y, the maps
$$M(X \sqcup Y) \underset{(M_*(\underline{i}_X),M_*(\underline{i}_Y))}{\overset{(M^*(\underline{i}_X),M^*(\underline{i}_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$
are inverse to each other, where $X \xhookrightarrow{i_X} X \sqcup Y \xhookleftarrow{i_Y} Y$.

2. if $\begin{array}{ccc} X & \stackrel{\underline{a}}{\to} & Y \\ {\scriptstyle \underline{b}}\downarrow & & \downarrow{\scriptstyle \underline{c}} \\ Z & \underset{\underline{d}}{\to} & T \end{array}$ is a weak pullback square in G-**set**

Fused Mackey functors

Definition

A fused Mackey functor is a bivariant functor G-**set**$\to \mathbb{Z}$-**Mod** such that:

1. for any finite G-sets X and Y, the maps
$$M(X \sqcup Y) \underset{(M_*(\underline{i}_X),M_*(\underline{i}_Y))}{\overset{(M^*(\underline{i}_X),M^*(\underline{i}_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$
are inverse to each other, where $X \overset{i_X}{\hookrightarrow} X \sqcup Y \overset{i_Y}{\hookleftarrow} Y$.
2. if $\begin{array}{ccc} X & \overset{a}{\to} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ Z & \underset{d}{\to} & T \end{array}$ is a weak pullback square in G-**set**, then
$$M^*(\underline{d}) \circ M_*(\underline{c}) = M_*(\underline{b}) \circ M^*(\underline{a}).$$

Fused Mackey functors

Definition

A fused Mackey functor is a bivariant functor G-**set**$\to \mathbb{Z}$-**Mod** such that:

1. for any finite G-sets X and Y, the maps
$$M(X \sqcup Y) \underset{(M_*(\underline{i}_X),M_*(\underline{i}_Y))}{\overset{(M^*(\underline{i}_X),M^*(\underline{i}_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$
are inverse to each other, where $X \xhookrightarrow{i_X} X \sqcup Y \xhookleftarrow{i_Y} Y$.

2. if $\begin{array}{ccc} X & \overset{\underline{a}}{\to} & Y \\ {\scriptstyle \underline{b}}\downarrow & & \downarrow{\scriptstyle \underline{c}} \\ Z & \underset{\underline{d}}{\to} & T \end{array}$ is a weak pullback square in G-**set**, then
$$M^*(\underline{d}) \circ M_*(\underline{c}) = M_*(\underline{b}) \circ M^*(\underline{a}).$$

Fused Mackey functors form a category $Mack^f(G)$.

Fused Mackey functors

Definition

A fused Mackey functor is a bivariant functor G-$\underline{\mathbf{set}} \to \mathbb{Z}$-$\mathbf{Mod}$ such that:

1. for any finite G-sets X and Y, the maps
$$M(X \sqcup Y) \underset{(M_*(\underline{i}_X),M_*(\underline{i}_Y))}{\overset{(M^*(\underline{i}_X),M^*(\underline{i}_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$
are inverse to each other, where $X \overset{i_X}{\hookrightarrow} X \sqcup Y \overset{i_Y}{\hookleftarrow} Y$.
2. if $\begin{array}{ccc} X & \overset{\underline{a}}{\to} & Y \\ {\scriptstyle \underline{b}}\downarrow & & \downarrow{\scriptstyle \underline{c}} \\ Z & \underset{\underline{d}}{\to} & T \end{array}$ is a weak pullback square in G-$\underline{\mathbf{set}}$, then
$$M^*(\underline{d}) \circ M_*(\underline{c}) = M_*(\underline{b}) \circ M^*(\underline{a}).$$

Fused Mackey functors form a category $Mack^f(G)$.

Theorem

Fused Mackey functors

Definition

A fused Mackey functor is a bivariant functor G-$\underline{\mathbf{set}} \to \mathbb{Z}$-**Mod** such that:

1. for any finite G-sets X and Y, the maps
$$M(X \sqcup Y) \underset{(M_*(\underline{i}_X), M_*(\underline{i}_Y))}{\overset{(M^*(\underline{i}_X), M^*(\underline{i}_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$
are inverse to each other, where $X \xhookrightarrow{i_X} X \sqcup Y \xhookleftarrow{i_Y} Y$.

2. if $\begin{array}{ccc} X & \xrightarrow{\underline{a}} & Y \\ {\scriptstyle \underline{b}}\downarrow & & \downarrow{\scriptstyle \underline{c}} \\ Z & \xrightarrow[\underline{d}]{} & T \end{array}$ is a weak pullback square in G-$\underline{\mathbf{set}}$, then
$$M^*(\underline{d}) \circ M_*(\underline{c}) = M_*(\underline{b}) \circ M^*(\underline{a}).$$

Fused Mackey functors form a category $Mack^f(G)$.

Theorem

The categories $Mack^c(G)$

Fused Mackey functors

Definition

A fused Mackey functor is a bivariant functor G-$\underline{\mathbf{set}} \to \mathbb{Z}$-**Mod** such that:

1. for any finite G-sets X and Y, the maps
$$M(X \sqcup Y) \underset{(M_*(\underline{i}_X), M_*(\underline{i}_Y))}{\overset{(M^*(\underline{i}_X), M^*(\underline{i}_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$
are inverse to each other, where $X \overset{i_X}{\hookrightarrow} X \sqcup Y \overset{i_Y}{\hookleftarrow} Y$.

2. if $\begin{array}{ccc} X & \overset{\underline{a}}{\to} & Y \\ {\scriptstyle \underline{b}}\downarrow & & \downarrow{\scriptstyle \underline{c}} \\ Z & \underset{\underline{d}}{\to} & T \end{array}$ is a weak pullback square in G-$\underline{\mathbf{set}}$, then
$$M^*(\underline{d}) \circ M_*(\underline{c}) = M_*(\underline{b}) \circ M^*(\underline{a}).$$

Fused Mackey functors form a category $Mack^f(G)$.

Theorem

The categories $Mack^c(G)$, $Mack^f(G)$

Fused Mackey functors

Definition

A fused Mackey functor is a bivariant functor G-$\underline{\mathbf{set}} \to \mathbb{Z}$-**Mod** such that:

1. for any finite G-sets X and Y, the maps
$$M(X \sqcup Y) \underset{(M_*(\underline{i}_X), M_*(\underline{i}_Y))}{\overset{(M^*(\underline{i}_X), M^*(\underline{i}_Y))}{\rightleftarrows}} M(X) \oplus M(Y)$$
are inverse to each other, where $X \overset{i_X}{\hookrightarrow} X \sqcup Y \overset{i_Y}{\hookleftarrow} Y$.
2. if $\begin{array}{ccc} X & \overset{a}{\to} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ Z & \underset{d}{\to} & T \end{array}$ is a weak pullback square in G-$\underline{\mathbf{set}}$, then
$$M^*(\underline{d}) \circ M_*(\underline{c}) = M_*(\underline{b}) \circ M^*(\underline{a}).$$

Fused Mackey functors form a category $Mack^f(G)$.

Theorem

The categories $Mack^c(G)$, $Mack^f(G)$, and $\mathcal{F}_G$ are equivalent.

The fused Mackey algebra

The fused Mackey algebra

Let $\Omega_G = \bigsqcup_{H \leq G} G/H$. Then:

The fused Mackey algebra

Let $\Omega_G = \bigsqcup_{H \leq G} G/H$. Then:

Theorem (Thévenaz-Webb, B.)

The category Mack(G) *is equivalent to the category* $\mu_{\mathbb{Z}}(G)$-**Mod**

The fused Mackey algebra

Let $\Omega_G = \bigsqcup_{H \leq G} G/H$. Then:

Theorem (Thévenaz-Webb, B.)

*The category $Mack(G)$ is equivalent to the category $\mu_{\mathbb{Z}}(G)$-**Mod**, where $\mu_{\mathbb{Z}}(G) = \mathcal{B}\big({}_G(\Omega_G \times \Omega_G)\big)$ is the Mackey algebra of G.*

The fused Mackey algebra

Let $\Omega_G = \bigsqcup_{H \leq G} G/H$. Then:

Theorem (Thévenaz-Webb, B.)

*The category $Mack(G)$ is equivalent to the category $\mu_{\mathbb{Z}}(G)$-**Mod**, where $\mu_{\mathbb{Z}}(G) = \mathcal{B}\big({}_G(\Omega_G \times \Omega_G)\big)$ is the Mackey algebra of G.*

Theorem

The fused Mackey algebra

Let $\Omega_G = \bigsqcup_{H \leq G} G/H$. Then:

Theorem (Thévenaz-Webb, B.)

*The category $Mack(G)$ is equivalent to the category $\mu_{\mathbb{Z}}(G)$-**Mod**, where $\mu_{\mathbb{Z}}(G) = \mathcal{B}\big({}_G(\Omega_G \times \Omega_G)\big)$ is the Mackey algebra of G.*

Theorem

Let $\mu^f_{\mathbb{Z}}(G)$ denote the fused Mackey algebra of G

The fused Mackey algebra

Let $\Omega_G = \bigsqcup_{H \leq G} G/H$. Then:

Theorem (Thévenaz-Webb, B.)

*The category $Mack(G)$ is equivalent to the category $\mu_{\mathbb{Z}}(G)$-**Mod**, where $\mu_{\mathbb{Z}}(G) = \mathcal{B}\big({}_G(\Omega_G \times \Omega_G)\big)$ is the Mackey algebra of G.*

Theorem

Let $\mu_{\mathbb{Z}}^f(G)$ denote the fused Mackey algebra of G, i.e. the quotient of $\mu_{\mathbb{Z}}(G) = \mathcal{B}\big({}_G(\Omega_G \times \Omega_G)\big)$ by the subgroup generated by differences

$$(*) \qquad \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "b"'] \arrow[dr, "a"] & \\ \Omega_G & & \Omega_G \end{tikzcd} \right) - \left(\begin{tikzcd}[column sep=small] & Z \arrow[dl, "u*b"'] \arrow[dr, "a"] & \\ \Omega_G & & \Omega_G \end{tikzcd} \right)$$

The fused Mackey algebra

Let $\Omega_G = \bigsqcup\limits_{H \leq G} G/H$. Then:

Theorem (Thévenaz-Webb, B.)

*The category Mack(G) is equivalent to the category $\mu_{\mathbb{Z}}(G)$-**Mod**, where $\mu_{\mathbb{Z}}(G) = \mathcal{B}\big({}_G(\Omega_G \times \Omega_G)\big)$ is the Mackey algebra of G.*

Theorem

Let $\mu_{\mathbb{Z}}^f(G)$ denote the fused Mackey algebra of G, i.e. the quotient of $\mu_{\mathbb{Z}}(G) = \mathcal{B}\big({}_G(\Omega_G \times \Omega_G)\big)$ by the subgroup generated by differences

$$(*) \qquad \left(\begin{array}{ccc} & Z & \\ {}^{b}\swarrow & & \searrow^{a} \\ \Omega_G & & \Omega_G \end{array}\right) - \left(\begin{array}{ccc} & Z & \\ {}^{u*b}\swarrow & & \searrow^{a} \\ \Omega_G & & \Omega_G \end{array}\right)$$

*Then the category Mack$^f(G)$ is equivalent to $\mu_{\mathbb{Z}}^f(G)$-**Mod**.*