HOMOLOGY COMPUTATIONS FOR COMPLEX BRAID GROUPS II

IVAN MARIN

Abstract. We complete the computation of the integral homology of the generalized braid group B associated to an arbitrary irreducible complex reflection group W of exceptional type. In order to do this we explicitely computed the recursively-defined differential of a resolution of $\mathbf{Z}$ as a $\mathbf{Z}B$-module, using parallel computing. We also deduce from this general computation the rational homology of the Milnor fiber of the singularity attached to most of these reflection groups.

1. Introduction

This paper is a sequel of [5], where we managed to compute the homology of many of the complex braid groups arising from complex reflection groups. Parts of our difficulties in completing the project were computational in nature. Since then, by using powerful computers during several months as well as a few computational tricks we managed to complete some of the tables. This paper is a report on these computations.

Recall from e.g. [2] that one can attach to every finite complex reflection group W a generalized braid group B. Without loss of generality, one can assume that W is irreducible. The usual classification, due to Shephard-Todd, of irreducible finite complex reflection groups, divides them into a general series depending on 3 parameters, and a finite set of 34 exceptions. It is therefore a natural question to explicitely compute the group homology of B when W belongs to this finite set of exceptional groups ; moreover, one knows that B has finite homological dimension by work of Bessis in [1].

By several arguments, recalled in [5], one can reduce the problem to a fewer number of groups. In particular, the homology of groups of small rank can be easily computed. Moreover, when W is a *real* reflection group, a general complex due to Salvetti (see [16]) can be used to compute the homology of B (see [16, 5]); in this case, B is an Artin group.

For these reasons, the remaining groups on which we need to focus are the ones labelled G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34} in Shephard and Todd notation. Complexes can be obtained from the so-called Garside theory introduced by Dehornoy and Paris. Indeed, Dehornoy and Lafont have proven in [7] that a Charney-Meyer-Wittlesey-type complex can be used whenever B is the group of fraction of a so-called Garside monoid. Bessis has proven that, when W is well-generated, then B satisfies this condition : there is one (and actually several) convenient Garside monoid(s) B^+ (see [1]). The one we use here has been specified for each group in [5]. All the groups above are well-generated, *but* G_{31}. Even in the case of G_{31}, one can define a similar complex, by attaching to B a so-called Garside category instead of a Garside monoid. However, as we noticed in [5], the complexes obtained by this method are very big, which poses a computational memory problem to compute their homology.

On the other hand, Dehornoy and Lafont introduced another, more mysterious but smaller kind of complex, which can be attached to a similar Garside monoid, but for which a generalization to Garside categories has not been proposed so far. Therefore, for this approach one needs to exclude the case of G_{31}. The drawback of this complex is that, while the computation of the homology of the complex is much easier as soon as it is explicitely described, the explicit computation of (the differential of) the complex itself is much more difficult and time-consuming. For the other groups of rank at least 3, the specific Garside monoids chosen for these groups have been specified in [5], table 7.

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In the present work, we computed this differential. The result is stored inside large files, and could in theory be used to compute the homology of $H_*(B,M)$ for an arbitrary $\mathbf{Z}B$-module M. In this paper, we describe the result of $H_*(B,M)$ in the following cases :

(1) $M = \mathbf{Z}$ with trivial action;
(2) $M = \mathbf{Z}$ with action given by the sign morphism $B \to W \stackrel{\det}{\to} \{\pm 1\}$, which exists because all the (pseudo-)reflections of W have order 2 in these cases;
(3) $M = \mathbf{Q}[t,t^{-1}]$ with action given by the natural map $B \to \mathbf{Z}$, $\sigma \mapsto t$.

Note that, when R is a commutative ring, $H_*(B, R[t,t^{-1}])$ can be identified with the homology of the Milnor fiber of the singularity corresponding to W (see [3]). For G_{34}, we were however unable to compute the homology of the Milnor fiber in ranks 4 to 6 because of computer and software limitations.

It seems likely that the Dehornoy-Lafont complex can be adapted to the kind of Garside categories that are suitable for dealing with G_{31}, using its description as a centralizer in the group of Coxeter type E_8, as in [1]. However such a theory has not been developped yet, and therefore G_{31} is, for the time being, out of reach of our computations.

As an indication of computing time, we mention that the computation, for G_{34}, of the differential of 2000 of the 7414 5-cells lasted around 200 days of CPU time on a SMP architecture. The computation of the differentials of the 5-cells and 6-cells altogether lasted around 78000 hours of CPU time.

Acknowledgements. The results presented here for G_{34} were obtained using the ressources of the MeCS platform of the Université de Picardie Jules Verne in Amiens, France. I thank very much Serge Van Criekingen for his help in using it. I also thank Filippo Callegaro for several discussions and comments.

2. Computational datas

The size of the complexes we had to compute are tabulated in table 1. Recall that each monoid B^+ is described as generated by a set A of atoms, and that there is a distinguished element Δ. Its set S of divisors is the same on the left and on the right, and is called the set of *simples* of the monoid B^+. We describe the apparatus for the case W is the complex reflection group of type G_{34}, also called Mitchell's group. Our programs and files are made to be primarily used by GAP3, but the syntax is quite standard whence these files could be used by probably every computer program with possibly only tiny changes to be made. These can be found at `http://www.lamfa.u-picardie.fr/marin/G34homology.html`.

The group W is described as a permutation group. The set A is in 1-1 correspondence with generators of B^+, which are stored in the file `atomsG34.gap` as an ordered list `allatoms` of 56 permutations. The set S is in 1-1 correspondence with a subset of the set of reflections of W, which is stored in the file `simplesG34.gap` as an ordered list `allsimples` of 1584 permutations. An important additional data is the length of each simple as a product of atoms. This data is stored, in the same file, as the list `simpleslengths`, in obvious bijection with the list `allsimples`, namely `simpleslengths[i]` is the length ot the simple element `allsimples[i]`.

By construction, the k-cells of the Dehornoy-Lafont complex are in 1-1 correspondence with lists of atoms of the form $[a_1,\dots,a_k]$ with $a_i \in A$. The files `cells2N.gap`,…,`cells6N.gap` store them, under the variable name `cells2N`,…,`cells6N`, as a list of lists $[c_1,\dots,c_k]$ so that a_i is the c_i-th atoms, namely c_i is the position in the list `allatoms` of the atom a_i. The 1-cells are simply given by the 56 atoms, and the differential is simply $\partial([a]) = a[\emptyset] - [\emptyset]$, where $[\emptyset]$ denotes the only 0-cell.

The files `Dcells2.gap`,…,`Dcells6.gap` contain, under the variable names `Dcells2P`,…,`Dcells6P`, the differentials of the k-cells, for $k \in \{2,\dots,6\}$. The format is as follows. For example, the variable `Dcells4P` is a list of 7520 elements $[v_1,\dots,v_{7520}]$, where v_r represents the differencial of the r-th cell in the list `cells4`. This differential is a linear combination of 3-cells with coefficients in the monoid algebra $\mathbf{Z}B^+$. This linear combination can be written $\sum_{i=1}^n a_i b_i c_i$, with $a_i \in \mathbf{Z}$, $b_i \in B^+$ and c_i a 3-cell.The element v_r is a list of n terms $[a_i, Q_i]$, where $Q_i = [\beta_i, \gamma_i]$ and β_i, γ_i

		0-cells	1-cells	2-cells	3-cells	4-cells	5-cells	6-cells
G_{24}	CMW	1	29	77	49			
	DL	1	14	38	25			
G_{27}	CMW	1	41	115	75			
	DL	1	20	62	43			
G_{29}	CMW	1	111	635	1025	500		
	DL	1	25	127	207	104		
G_{33}	CMW	1	307	3249	9747	11178	4374	
	DL	1	30	226	638	740	299	
G_{34}	CMW	1	1583	31717	163219	337169	304927	100842
	DL	1	56	711	3448	7520	7414	2686

TABLE 1. Compared size of the complexes

	H_0	H_1	H_2	H_3	H_4	H_5	H_6
G_{12}	$\mathbf{Z}$	$\mathbf{Z}$	0				
G_{13}	$\mathbf{Z}$	$\mathbf{Z}^2$	$\mathbf{Z}$				
G_{22}	$\mathbf{Z}$	$\mathbf{Z}$	0				
G_{24}	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$			
G_{27}	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_3 \times \mathbf{Z}$	$\mathbf{Z}$			
G_{29}	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_2 \times \mathbf{Z}_4$	$\mathbf{Z}_2 \times \mathbf{Z}$	$\mathbf{Z}$		
G_{31}	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_6$	$\mathbf{Z}$	$\mathbf{Z}$		
G_{33}	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_6$	$\mathbf{Z}_6$	$\mathbf{Z}$	$\mathbf{Z}$	
G_{34}	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_6$	$\mathbf{Z}_6$	$\mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_6$	$\mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}$	$\mathbf{Z}$

TABLE 2. Homology with trivial integer coefficients

	H_0	H_1	H_2	H_3	H_4	H_5	H_6
G_{12}	$\mathbf{Z}_2$	$\mathbf{Z}_3$	0				
G_{13}	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}$				
G_{22}	$\mathbf{Z}_2$	0	0				
G_{24}	$\mathbf{Z}_2$	0	$\mathbf{Z}_2$	0			
G_{27}	$\mathbf{Z}_2$	0	$\mathbf{Z}_2$	0			
G_{29}	$\mathbf{Z}_2$	0	$\mathbf{Z}_2 \times \mathbf{Z}_4$	$\mathbf{Z}_2 \times \mathbf{Z}_{40}$	0		
G_{33}	$\mathbf{Z}_2$	0	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	0	
G_{34}	$\mathbf{Z}_2$	0	$\mathbf{Z}_6$	$\mathbf{Z}_2$	$\mathbf{Z}_6$	$\mathbf{Z}_{252}$	0

TABLE 3. Homology with coefficients in the integer sign representation

encode b_i, c_i. The encoding is as follows. An element of B^+ is a product of simple elements $s_1, \ldots, s_q$. Then, β_i is a list $[\sigma_1, \ldots, \sigma_q]$ where σ_j is the position of s_j in the liste `allsimples`. Finally, γ_i encodes the cell c_i as in the file `cells3N.gap` (that is, as a list of positions in the atom's list `allatoms`).

3. THE RESULTS

3.1. **Hand-made computations for groups of rank 2.** For the groups G_{12} and G_{22} we use the monoids

$$B_{12}^+ = \langle x_1, x_2, x_3 \mid x_1x_2x_3x_1 = x_2x_3x_1x_2 = x_3x_1x_2x_3 \rangle$$
$$B_{22}^+ = \langle x_1, x_2, x_3 \mid x_1x_2x_3x_1x_2 = x_2x_3x_1x_2x_3 = x_3x_1x_2x_3x_1 \rangle$$

It is proved in [15] (see ch. 5 ex. 11 and ch. 6) that these monoids are Garside monoids, so we can build a Dehornoy-Lafont complex for each one of them. In both cases, the set of atoms is $\mathcal{X} = \{x_1, x_2, x_3\}$, which we endow with the linear ordering $x_1 < x_2 < x_3$. The 2-cells are then $[x_1, x_2]$ and $[x_1, x_3]$ and there are no cells of higher degree. Therefore the only datas to be computed in order to define the corresponding chain complex are $\partial_2[x_1, x_2]$ and $\partial_2[x_1, x_3]$. For G_{12} one gets

$$\begin{array}{rcl} \partial_2[x_1,x_2] & = & x_2x_3x_1[x_2] - x_1x_2x_3[x_1] - x_1x_2[x_3] - x_1[x_2] - [x_1] + x_2x_3[x_1] + x_2[x_3] + [x_2] \\ \partial_2[x_1,x_3] & = & x_3x_1x_2[x_3] - x_1x_2x_3[x_1] - x_1x_2[x_3] - x_1[x_2] - [x_1] + x_3x_1[x_2] + x_3[x_1] + [x_3] \end{array}$$

and for G_{22} one gets

$$\begin{array}{rcl} \partial_2[x_1,x_2] & = & x_1x_2x_3x_1[x_2] - x_3x_1x_2x_3[x_1] - x_3x_1x_2[x_3] - x_3x_1[x_2] - x_3[x_1] - [x_3] + x_1x_2x_3[x_1] \\ & & +x_1x_2[x_3] + x_1[x_2] + [x_1] \\ \partial_2[x_1,x_3] & = & x_2x_3x_1x_2[x_3] - x_3x_1x_2x_3[x_1] - x_3x_1x_2[x_3] - x_3x_1[x_2] - x_3[x_1] - [x_3] + x_2x_3x_1[x_2] \\ & & +x_2x_3[x_1] + x_2[x_3] + [x_2] \end{array}$$

The $\mathbf{Z}B$-module structure on $\mathbf{Q}[t, t^{-1}]$ affording the homology of the Milnor fiber is given by $x_i \mapsto t$ in both cases (see [4, 3] for the classical case).

For the group G_{13}, we use the fact that its braid group is isomorphic to the Artin group of type $I_2(6)$. More precisely, in this case B has for presentation $\langle x, y, z \mid yzxy = zxyz, zxyzx = xyzxy \rangle$ and an isomorphism with the Artin group $\langle a, b \mid ababab = bababa \rangle$ is given by the formulas

$$\left\{ \begin{array}{rcl} a & = & zx \\ b & = & zxy(zx)^{-1} \end{array} \right. \qquad \left\{ \begin{array}{rcl} x & = & (baba)^{-1}abaa \\ y & = & a^{-1}ba \\ z & = & (aba)^{-1}b(aba) \end{array} \right.$$

In particular we have $B^{ab} \simeq \mathbf{Z}^2$, and two different bases are afforded by $(\bar{x}, \bar{y} = \bar{z})$ and $(\bar{a}, \bar{b})$, the relation between them being $\bar{x} = \bar{a} - \bar{b}$, $\bar{y} = \bar{b}$. The (homological) Salvetti complex (see [6]) is made of free $\mathbf{Z}B$-modules with basis $[\emptyset]$ (one 0-cell), $[a], [b]$ (two 1-cells) and $[a, b]$ (one 2-cells). The differential is given by $\partial[\emptyset] = 0$, $\partial[a] = (a-1)[\emptyset]$, $\partial[b] = (b-1)[\emptyset]$, and

$$\partial[a,b] = (1 - a + ab - aba + abab - ababa)[b] - (1 - b + ba - bab + baba - babab)[a].$$

Therefore, tensoring by the $\mathbf{Z}B$-module $R = \mathbf{Z}[t, t^{-1}]$-module defined by $x, y, z \mapsto t$, we get the differential $d[a] = (t^2 - 1)[\emptyset]$, $d[b] = (t-1)[\emptyset]$, $d[a, b] = (1 + t^3 + t^6)(1 - t)\left((1 + t)[b] - [a]\right)$. Specializing at $t = 1$ and $t = -1$ we readily get the results of tables 2 and 3 for G_{13}. In general, one gets the following homology of the Milnor fiber for G_{13} :

$$H_0 = R/(t-1)R \simeq \mathbf{Z}, H_1 = R/(1-t)(1+t^3+t^6) \simeq \mathbf{Z}^6, H_2 = 0.$$

3.2. **Milnor fibers.** We let $\Phi_n \in \mathbf{Z}[t]$ denote the n-th (rational) cyclotomic polynomial, and $R = \mathbf{Q}[t, t^{-1}]$. In table 4 we indicate the homology $H_*(B, \mathbf{Q}[t, t^{-1}])$ as a R-module, which can be identified with the rational homology of the Milnor fiber. Results on the classical (Artin) cases can be found in [3, 4]. In our paper the RB-module structure on $\mathbf{Q}[t, t^{-1}]$ is given by $\sigma_i \mapsto t$ (while the choice in [4] is $\sigma_i \mapsto -q$). This homology was computed using the software `Macaulay2`, see [9]. In the table, for each $P \in R$, the presence of P in the table symbolizes the R-module $R/(P)$, and $\mathbf{Q}$ is a shortcut for $R/\Phi_1 = \mathbf{Q}[t, t^{-1}]/(t-1)$, that we use for H_0 and H_1. Notice that

$$\frac{t^{20}-1}{t+1} = \frac{t^{20}-1}{\Phi_2} = \Phi_1 \oplus \Phi_4 \oplus \Phi_5 \oplus \Phi_{10} \oplus \Phi_{20}$$

	H_0	H_1	H_2	H_3	H_4	H_5	H_6
G_{12}	$\mathbf{Q}$	$\mathbf{Q}$	$\Phi_6 \oplus \Phi_{12}$				
G_{13}	$\mathbf{Q}$	$\mathbf{Q} \oplus \Phi_9$	0				
G_{22}	$\mathbf{Q}$	$\mathbf{Q}$	Φ_{15}				
G_{24}	$\mathbf{Q}$	$\mathbf{Q}$	0	$\Phi_1 \oplus \Phi_3 \oplus \Phi_7$			
G_{27}	$\mathbf{Q}$	$\mathbf{Q}$	0	$(t^{15}-1) \oplus \Phi_3$			
G_{29}	$\mathbf{Q}$	$\mathbf{Q}$	0	$\Phi_4 \oplus \Phi_4$	$\frac{t^{20}-1}{t+1} \oplus \Phi_4$		
G_{33}	$\mathbf{Q}$	$\mathbf{Q}$	0	0	0	$(t^9-1) \oplus \Phi_5$	
G_{34}	$\mathbf{Q}$	$\mathbf{Q}$	0	Φ_6	?	?	?

TABLE 4. Rational homology $H_*(B, \mathbf{Q}[t,t^{-1}])$ of the Milnor fiber

It follows from the table that the Poincaré polynomials of the Milnor fibers are as follows.

Group	Poincaré polynomial
G_{12}	$1+x+6x^2$
G_{13}	$1+6x$
G_{22}	$1+x+8x^2$
G_{24}	$1+x+9x^3$
G_{27}	$1+x+17x^3$
G_{29}	$1+x+4x^3+21x^4$
G_{33}	$1+x+13x^5$
G_{34}	$1+x+2x^3+\ldots?$

With the same algorithm, for any given p we can compute the homology $H_*(B, \mathbf{F}_p[t,t^{-1}])$ modulo p of the Milnor fiber. We compute this for $p \in \{2,3,5,7\}$. These numbers might be particularly interesting for applications because they are the only primes dividing $|W|$ for W in our list. Letting Φ_n denote the reduction modulo p of the n-th usual (rational) cyclotomic polynomial, we get the same result as in table 4, *except* for the following cases :

- When $W = G_{12}$ and $p \in \{2,3,5,7\}$ we have $H_2(B, \mathbf{F}_p[t,t^{-1}]) = P_{12,2}$ with $P_{12,2} = t^6 - t^5 + t^3 - t + 1$. When $p = 3,5,7$ this amounts to $\Phi_6 \oplus \Phi_{12}$ as in the rational case, but in case $p = 2$ we have $P_{12,2} = (t^2+t+1)^3 = \Phi_3^3$.
- When $W = G_{13}$ and $p \in \{2,3,5,7\}$ we have $H_1(B, \mathbf{F}_p[t,t^{-1}]) = P_{13,1}$ with $P_{13,1} = (1-t)(1+t^3+t^6)$. When $p = 2,5,7$ this amounts to $\Phi_1 \oplus \Phi_9$ as in the rational case, but in case $p = 3$ we have $P_{13,1} = (t-1)^7$.
- When $W = G_{24}$, and $p \in \{2,3,5,7\}$, we have $H_3(B, \mathbf{F}_p[t,t^{-1}]) = P_{24,3}$ with $P_{24,3}(t) = t^9+t^8+t^7-t^2-t-1$. When $p = 2,5$ this amounts to $\Phi_1 \oplus \Phi_3 \oplus \Phi_7$ as in the rational case, but in case $p = 3$ we have $P_{24,3} = (t-1)^3\Phi_7$ while for $p = 7$ we have $P_{24,3} = (t+3)(t-1)^7(t+5)$.
- When $W = G_{29}$, and $p = 2$, we have $H_3(B, \mathbf{F}_p[t,t^{-1}]) = (t+1)^3 \oplus \Phi_4$ (instead of $\Phi_4 \oplus \Phi_4$).
- When $W = G_{33}$, and $p = 2,3$, we have $H_3(B, \mathbf{F}_p[t,t^{-1}]) = \Phi_1$ (instead of 0).
- When $W = G_{34}$, and $p = 2,3$, we have $H_3(B, \mathbf{F}_p[t,t^{-1}]) = \Phi_6 \oplus \Phi_1$ (instead of Φ_6).
- When $W = G_{29}$, and $p = 2$, we have $H_4(B, \mathbf{F}_2[t,t^{-1}]) = (t^{20}-1) \oplus \Phi_4$ (instead of $\frac{t^{20}-1}{t+1} \oplus \Phi_4$).
- When $W = G_{33}$, and $p = 2,3$, we have $H_4(B, \mathbf{F}_p[t,t^{-1}]) = \Phi_1$ (instead of 0).
- When $W = G_{33}$, and $p \in \{2,3,5,7\}$, we have $H_5(B, \mathbf{F}_p[t,t^{-1}]) = P_{33,5}$ with $P_{33,5} = t^{13}+t^{12}+t^{11}+t^{10}+t^9-t^4-t^3-t^2-t-1$. When $p = 2,3,7$ this amounts to $(t^9-1) \oplus \Phi_5$ as in the rational case, but when $p = 5$ we have $P_{33,5} = (t-1)^5(t^2+t+1)(t^6+t^3+1)$.

F. Callegaro observed and communicated to us that some of the special cases we obtain here can be unified by using that $\Phi_{mp^i}(t) = \Phi_m(t)^{\phi(p^i)} \mod p$ (see e.g. [10]). Moreover, it follows from these computations that there are no torsion of order $2,3,5,7$ in the integer homology of the Milnor fiber for $G_{12}, G_{13}, G_{22}, G_{24}, G_{27}$, while there is 2-torsion for G_{29} and 2,3-torsion for G_{33}.

We were not able to compute the higher dimensional rational homology groups of the Milnor fiber for G_{34}, because of memory issues with the software we are using. However, someone having more computational skills than us might well be able to compute them, using smarter procedures. Therefore, we included the `Macaulay2` script among the datafiles which can be found on our webpage.

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LAMFA, Université de Picardie-Jules Verne, Amiens, France
E-mail address: `ivan.marin@u-picardie.fr`