

ON THE LARGEST REPRESENTATION IN TYPE H_4

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To the memory of Kay Magaard

ABSTRACT. In this technical note, we complete the PhD work of A. Esterle about determining the image of any Artin group of finite Coxeter type inside the associated Hecke algebra over a finite field, when the latter is semisimple. The only remaining case was the 48-dimensional irreducible representation in type H_4 , for which the image is proven here to be Ω_{48}^+ .

1. INTRODUCTION

In [2], A. Esterle determined the image of the Artin groups of finite Coxeter type inside the corresponding Hecke algebra over a finite field, under some conditions on the parameter essentially made so that the latter is semisimple. There is just *one* case for which the methods of [2] were not conclusive, because a main technical tool due to Guralnick and Saxl could not be applied there. This case is the largest irreducible representation of the exceptional Coxeter type H_4 . Here we solve this remaining case by going back to the original basic tool in this area, namely Aschbacher's classification theorem.

We use the notational conventions of A. Esterle's thesis [2]. In particular, as in [2], let us denote A_{H_4} the Artin group of type H_4 and $\mathcal{A}_{H_4}$ its derived subgroup. We consider the representation $\rho : \mathcal{A}_{H_4} \rightarrow \mathrm{GL}_{48}(\overline{\mathbb{F}_p})$ deduced from the only 48-dimensional irreducible representation of the Hecke algebra in type H_4 , and we want to determine the image G of ρ . The convention on Hecke algebras is that the Artin generators have eigenvalues $-1, \alpha$.

It is the only unsolved case in [2], where it is shown that G is a subgroup of $\Omega_{48}^+(\sqrt{q})$, depending on the parameter $\alpha \in \mathbb{F}_q^\times$ with $\mathbb{F}_q = \mathbb{F}_p(\alpha)$. It is assumed that $p \notin \{2, 3, 5\}$ and the order of α in $\mathbb{F}_q^\times$ is assumed not to divide 20, 30 or 48. These conditions imply $q \geq 19$ and, when q is a square, that $q \geq 121$. Moreover, since the other cases are already dealt with in [2], one can assume that (condition '1 ~ 2' in [2]) there exists a field automorphism Φ of $\mathbb{F}_p(\alpha, \xi + \xi^{-1})$ such that $\Phi(\alpha + \alpha^{-1}) = \alpha + \alpha^{-1}$ and $\Phi(\xi + \xi^{-1}) = \xi^2 + \xi^{-2}$, which is equivalent ([2] lemma 8.1.2) to the condition $\mathbb{F}_p(\alpha, \xi + \xi^{-1}) = \mathbb{F}_p(\alpha + \alpha^{-1}, \xi + \xi^{-1})$. We set $\mathbb{F}_g = \mathbb{F}_p(\alpha + \alpha^{-1})$.

Because of the existence of $\Phi \in \mathrm{Gal}(\mathbb{F}_g(\xi + \xi^{-1})/\mathbb{F}_g) \setminus \{\mathrm{Id}\}$, $\mathbb{F}_g$ has index 2 inside $\mathbb{F}_g(\xi + \xi^{-1})$. Now, it is proven in [2] that the representation of $\mathcal{A}_{H_4}$ under consideration admits a matrix model over $\mathbb{F}_g(\xi + \xi^{-1})$, so we can write it as $\rho : \mathcal{A}_{H_4} \rightarrow \mathrm{GL}_{48}(\mathbb{F}_g(\xi + \xi^{-1}))$. If $\mathbb{F}_p(\alpha) = \mathbb{F}_g$, that is $q = g$, since there is only one 48-dimensional irreducible representation of A_{H_4} that factorizes through the Hecke algebra of type H_4 , and since this one is defined over $\mathbb{F}_g$, we have $\Phi \circ \rho \simeq \rho$ and we can assume that ρ takes value inside $\mathrm{GL}_{48}(\mathbb{F}_g) = \mathrm{GL}_{48}(\mathbb{F}_q)$ (see

e.g. [2] lemma 3.2.5). If not, we have that $[\mathbb{F}_p(\alpha) : \mathbb{F}_g] = 2$ hence $\mathbb{F}_p(\alpha) = \mathbb{F}_g(\xi + \xi^{-1})$ and ρ also takes value inside $\mathrm{GL}_{48}(\mathbb{F}_q)$, this time with $q = g^2$. But then Φ is the only non-trivial element of $\mathrm{Gal}(\mathbb{F}_p(\alpha)/\mathbb{F}_g)$, hence exchanges α and α^{-1} . From Proposition 2.1.2 of [2] this implies that $\Phi \circ \rho \simeq \rho$ hence ρ can actually be defined over $\mathbb{F}_p(\alpha + \alpha^{-1}) = \mathbb{F}_g$ and this proves that ρ can be defined with values in $\mathrm{GL}_{48}(\mathbb{F}_g)$ again.

The purpose of this note is to establish the following statement, which solves conjecture 8.5.1 of [2] and completes the goal of A. Esterle thesis. Notice that actually the case $\mathbb{F}_p(\alpha) = \mathbb{F}_p(\alpha + \alpha^{-1})$ was overlooked there, which makes Conjecture 8.5.1 trivially false as stated in this case, so that Theorem 8.5.1 of [2] needs to be corrected. Explicitely, in the cases 1.a, 1.d of the theorem, the factor $\Omega_{48}^+(\sqrt{q})$ of the decomposition must be replaced by $\Omega_{48}^+(q)$.

Theorem 1.1. *In the case 1 ~ 2, and with the notations of [2], we have $G = \Omega_{48}^+(g)$, with $\mathbb{F}_g = \mathbb{F}_p(\alpha + \alpha^{-1})$.*

Distinguishing cases, this statement can be rewritten as follows.

- (1) If $\mathbb{F}_q = \mathbb{F}_p(\alpha) = \mathbb{F}_p(\alpha + \alpha^{-1})$, then $G = \Omega_{48}^+(q)$
- (2) If $\mathbb{F}_q = \mathbb{F}_p(\alpha) \neq \mathbb{F}_p(\alpha + \alpha^{-1}) = \mathbb{F}_{\sqrt{q}}$, then $G = \Omega_{48}^+(\sqrt{q})$

As explained in [2], this case could not be tackled by the methods used there, because the restrictions to some parabolic subgroup do not contain elements of the type needed for applying a very handy theorem of Guralnick and Saxl. Therefore we change strategy and directly apply Aschbacher's theorem (see [1]). By the considerations above this theorem implies that it is sufficient to show that G cannot belong to any of the Aschbacher classes $\mathcal{C}_1, \dots, \mathcal{C}_7$ and $\mathcal{S}$.

By [2] theorem 8.1.1 p. 166 and the fact that the representation 48_{rr} of the Coxeter group of type H_4 restricts to $3_s + 3'_s + \overline{3_s} + \overline{3'_s} + 2 \times 4_r + 2 \times 4'_r + 2 \times 5_r + 2 \times 5'_r$ as a representation of type H_3 (see [2] Table 8.3), we know that G contains $Q_3 \times Q_4 \times Q_5$ with $Q_4 \in \{\mathrm{SL}_4(q), \mathrm{SU}_4(\sqrt{q})\}$ and $(Q_3, Q_5) \in \{(\mathrm{SL}_3(q^2), \mathrm{SL}_5(q)), (\mathrm{SL}_3(q), \mathrm{SU}_5(\sqrt{q}))\}$ embedded inside $\mathrm{SL}_{48}(\mathbb{F}_q(\xi + \xi^{-1}))$ via

$$(q_3, q_4, q_5) \mapsto (q_3, \bar{q}_3, {}^t q_3^{-1}, {}^t \bar{q}_3^{-1}, q_4, q_4, {}^t q_4^{-1}, {}^t \bar{q}_4^{-1}, q_5, q_5, {}^t q_5^{-1}, {}^t \bar{q}_5^{-1})$$

where $\bar{x} = \Phi(x)$. We also know that this subgroup normally generates G ([2] Lemma 8.2.1, p. 170). We have $|Q_3| \geq \max(|\mathrm{SL}_3(19)|, |\mathrm{SL}_3(19^2)|) \geq 1.5 \times 10^{19}$, $|Q_4| \geq \max(|\mathrm{SL}_4(19)|, |\mathrm{SU}_4(11)|) \geq 4 \times 10^{15}$, $|Q_5| \geq \max(|\mathrm{SL}_5(19)|, |\mathrm{SU}_5(11)|) \geq 9.7 \times 10^{24}$. We also notice that the order of the p -Sylow subgroups of Q_3, Q_4, Q_5 are at least $q^3, \sqrt{q}^6$ or q^6 , and $\sqrt{q}^{10}$ or q^{10} , respectively.

When one needs to distinguish the two cases, we excerpt again from [2] Theorem 8.1.1 that, when $\mathbb{F}_q = \mathbb{F}_p(\alpha) = \mathbb{F}_p(\alpha + \alpha^{-1})$, then $(Q_3, Q_5) = (\mathrm{SL}_3(q^2), \mathrm{SL}_5(q))$ and when $\mathbb{F}_q = \mathbb{F}_p(\alpha) \neq \mathbb{F}_p(\alpha + \alpha^{-1}) = \mathbb{F}_{\sqrt{q}}$, then $(Q_3, Q_5) = (\mathrm{SL}_3(q), \mathrm{SU}_5(\sqrt{q}))$.

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2. CLASSICAL ASCHBACHER CLASSES

Assume G is contained in a maximal subgroup Γ of $\mathrm{GL}_{48}(r)$ belonging to Aschbacher's class $\mathcal{C}_i$ with $1 \leq i \leq 7$ as in [1]. Class $\mathcal{C}_1$ is excluded because G acts absolutely irreducibly ([2], proposition 8.2.2). Similarly, classes $\mathcal{C}_6$ and $\mathcal{C}_7$ are excluded because 48 is not a non-trivial power. We thus need to consider classes $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5$.

2.1. Class $\mathcal{C}_2$. We assume now that Γ lies in class $\mathcal{C}_2$, that is $\mathbb{F}_r^{48} = V = V_1 \oplus \cdots \oplus V_s$ with $\dim V_1 = \cdots = \dim V_s = v$ so that $48 = vs$, and $\Gamma = \mathrm{GL}_v(r) \wr \mathfrak{S}_s$ with $v, s \geq 2$. Let $\pi : \Gamma \rightarrow \mathfrak{S}_s$ denote the associated projection and consider the map $\pi_k : Q_k \rightarrow \mathfrak{S}_s$ obtained by composing π with $Q_k \rightarrow Q_3 \times Q_4 \times Q_5 \rightarrow G$. Since Q_k is almost simple, either $\mathrm{Ker} \pi_k < Z(Q_k)$ or $\mathrm{Ker} \pi_k = Q_k$. In the first case, π_k induces an embedding of the p -Sylow subgroup of Q_k inside $\mathfrak{S}_s$. But since $p \geq 11$ and s is a proper divisor of 48, an immediate computation shows that the orders of the p -Sylow subgroups of $\mathfrak{S}_s$ for such s are at most p^2 , contradicting our conditions. It follows that $\pi(G) = \{1\}$ hence G acts non-irreducibly on V , a contradiction.

2.2. Class $\mathcal{C}_3$. We have $\Gamma < \mathrm{SL}_{48}(\mathbb{F}_g)$. If Γ belongs to the class $\mathcal{C}_3$, then we can write $\mathbb{F}_g^{48} = \mathbb{F}_s^h$ for $h < 48$, with $\mathbb{F}_s \subsetneq \mathbb{F}_g$, and $\Gamma < \mathrm{SL}_h(\mathbb{F}_s).C_m$ for some cyclic group C_m . Since $Q_3 \times Q_4 \times Q_5$ is perfect, its image inside C_m is trivial, whence its image inside $G < \Gamma$ lies inside $\mathrm{SL}_h(\mathbb{F}_s)$. Since it normally generates G , we get that $G < \mathrm{SL}_h(\mathbb{F}_s)$. Since the 48-dimensional representation of G is absolutely irreducible we get a contradiction.

2.3. Class $\mathcal{C}_4$. We want to prove that the action of G is tensor-indecomposable. For this it is sufficient to prove that the action of $Q_3 < G$ is tensor-indecomposable. We thus consider $W = V \otimes_{\mathbb{F}_r} \overline{\mathbb{F}}_p = \overline{\mathbb{F}}_p^{48}$ as a Q_3 -module, and assume it can be written as $W_1 \otimes W_2$. It is a semisimple module, that can be decomposed as $U + \overline{U} + U^* + \overline{U}^* + 36 \cdot \mathbb{1}$ where $U, \overline{U}, U^*, \overline{U}^*$ are four absolutely irreducible pairwise non-isomorphic 3-dimensional representations of Q_3 . Since the only prime divisors of 48 are 2 and 3 we have $48 \not\equiv 0 \pmod p$ hence W_1 and W_2 are semisimple by [8] Theorem 2.4. We write a direct sum decomposition $W_k = m_k \mathbb{1} + \sum_{i=1}^{r_k} a_i^{(k)} F_i^{(k)}$ where the $F_i^{(k)}$ are pairwise nonisomorphic nontrivial simple modules. Since Q_3 is perfect they all have dimension at least 2. Then, the fact that the nontrivial irreducible constituents of W appear multiplicity free implies $m_1, m_2 \in \{0, 1\}$. We have

$$W = W_1 \otimes W_2 = m_1 m_2 \mathbb{1} + m_1 \sum_{i=1}^{r_2} a_i^{(2)} F_i^{(2)} + m_2 \sum_{i=1}^{r_1} a_i^{(1)} F_i^{(1)} + \sum_{i=1}^{r_1} \sum_{k=1}^{r_2} a_i^{(1)} a_j^{(2)} F_i^{(1)} \otimes F_j^{(2)}$$

and, since W contains $36 \cdot \mathbb{1}$ and $F_i^{(1)} \otimes F_i^{(2)}$ may contain the trivial representation at most once by Schur's lemma, this implies

$$m_1 m_2 + \left(\sum_{i=1}^{r_1} a_i^{(1)} \right) \left(\sum_{i=1}^{r_2} a_i^{(2)} \right) \geq 36$$

and $m_1 m_2 \leq 1$ implies $\left(\sum_{i=1}^{r_1} a_i^{(1)} \right) \left(\sum_{i=1}^{r_2} a_i^{(2)} \right) \geq 35$. We have $\dim F_i^{(k)} \geq 2$ for all i, k since Q_3 is perfect hence $\mathbb{1}$ is the only 1-dimensional representation. Then $48 = (\dim W_1)(\dim W_2)$ and $\dim W_k = m_k + \sum_{i=1}^{r_k} a_i^{(k)} \dim F_i^{(k)} \geq 2 \sum_{i=1}^{r_k} a_i^{(k)}$ thus $48 = \dim W \geq 4 \times 35$, a contradiction. Therefore it is tensor-indecomposable.

2.4. Class $\mathcal{C}_5$. We have $\Gamma < \mathrm{SL}_{48}(\mathbb{F}_g)$. If Γ belongs to the class $\mathcal{C}_5$, then we can write $\mathbb{F}_g^{48} = \mathbb{F}_s^{48} \otimes_{\mathbb{F}_s} \mathbb{F}$ with $\mathbb{F}_s \subsetneq \mathbb{F}$, and $\Gamma < D = \mathrm{SL}_{48}(\mathbb{F}_s).(\mathbb{F}_g)^\times$. Since $D/\mathrm{SL}_{48}(\mathbb{F}_s) \simeq (\mathbb{F}_g)^\times/(\mathbb{F}_s)^\times$ is a cyclic group and $Q_3 \times Q_4 \times Q_5$ is perfect, its image inside $D/\mathrm{SL}_{48}(\mathbb{F}_s)$ is trivial, whence its image inside $G < \Gamma$ lies inside $\mathrm{SL}_{48}(\mathbb{F}_s)$. Since it normally generates G , we get that $G < \mathrm{SL}_{48}(\mathbb{F}_s)$.

We then distinguish two cases. First assume that $\mathbb{F}_q = \mathbb{F}_p(\alpha) \neq \mathbb{F}_p(\alpha + \alpha^{-1}) = \mathbb{F}_{\sqrt{q}} = \mathbb{F}_g$. In that case, from [2] theorem 8.1.1 (p.166) we get that $Q_3 = \mathrm{SL}_3(q)$. Let $q_3 =$

$\text{diag}(\alpha, \alpha^{-1}, 1) \in Q_3$, $q_4 = 1$, $q_5 = 1$. The image of (q_3, q_4, q_5) inside G has trace $2(\alpha + \alpha^{-1}) + 2\Phi(\alpha + \alpha^{-1}) + 40 = 4(\alpha + \alpha^{-1}) + 40$. It follows that $\mathbb{F}_s \supset \mathbb{F}_p(\alpha + \alpha^{-1}) = \mathbb{F}_{\sqrt{q}}$ hence $s = \sqrt{q}$, contradicting $\mathbb{F}_s \not\subseteq \mathbb{F}_{\sqrt{q}}$.

We then assume $\mathbb{F}_q = \mathbb{F}_p(\alpha) = \mathbb{F}_p(\alpha + \alpha^{-1}) = \mathbb{F}_g$. In that case, from [2] theorem 8.1.1 (p.166) we get that $Q_3 = \text{SL}_3(q^2)$ and $Q_5 = \text{SL}_5(q)$. Letting $q_5 = \text{diag}(\alpha, \alpha^{-1}, 1, 1, 1)$ and $q_3 = 1$, $q_4 = 1$, we get that the image of (q_3, q_4, q_5) inside G has trace $4(\alpha + \alpha^{-1}) + 40$ hence $\mathbb{F}_s \supset \mathbb{F}_p(\alpha + \alpha^{-1}) = \mathbb{F}_q$, contradicting $\mathbb{F}_s \not\subseteq \mathbb{F}_q$.

3. ASCHBACHER CLASS $\mathcal{S}$

We finally exclude the groups in the $\mathcal{S}$ class.

3.1. Groups of Lie type in non-defining characteristic and sporadic groups. We use [3] in this part where all the cases in small dimension have been classified.

	Γ	ℓ	field	$ \Gamma $
(1)	$2.\mathfrak{A}_8$	$\neq 2$		40320
(2)	$\mathfrak{A}_9$	$0, 2$		181440
(3)	$2.\mathfrak{A}_9$	3		362880
(4)	$2.\mathfrak{A}_9$	$\neq 2, 3$	$i6$	362880
(5)	$\mathfrak{A}_{10}$	2		1814400
(6)	$2.\mathfrak{A}_{10}$	3		3628800
(7)	$2.\mathfrak{A}_{10}$	$\neq 2, 3$	$i6$	3628800
(8)	$12_1.L_3(4)$	$0, 7$	$z12, b5$	241920
(9)	$12_2.L_3(4)$	$0, 7$	$z12, b5$	241920
(10)	$12_2.L_3(4)$	5	$z12$	241920
(11)	$3.U_3(5)$	$\neq 3, 5$	$z3$	378000
(12)	$2.S_6(2)$	$\neq 2, 7$		2903040
(13)	$O_8^+(2)$	3		174182400
(14)	$2.Sz(8)$	5	$c13$	58240
(15)	$12.M_{22}$	5	$z12, b11$	5322240

This table is the relevant excerpt of exceptional cases from table 3 in [3] (corrected as table 2 in [4]), completed by the order of the group. Since $|\Gamma| \geq |G| \geq |Q_3| \times |Q_4| \times |Q_5| \geq 58 \times 10^{58}$, the computation of this order is sufficient to dismiss these cases.

The generic cases explained in [3] (table 2 there) are of two types. One of these types (cases (b),(c),(d) in [3], table 2) is when G is either $2.PSL_2(m)$ or $PSL_2(m)$, with $m \leq 2 \times 48 + 1 \leq 100$. But then

$$|G| \leq 2 \times 100 \times (100^2 - 1) \leq 2 \times 10^6$$

contradicting $|G| \geq 58 \times 10^{58}$. The other type (case (a) in [3], table 2) is when $G = \mathfrak{A}_n$ with $n \in \{49, 50\}$. Then, since $p \geq 7$,

$$v_p(|G|) \leq v_p(50!) = \left\lfloor \frac{50}{p^2} \right\rfloor + \left\lfloor \frac{50}{p} \right\rfloor \leq 2 + 8 = 10$$

while $Q_3 \times Q_4 \times Q_5$ has a p -Sylow of order at least p^{19} , dismissing this case again.

Therefore this case is excluded, too.

3.2. Groups of Lie type in natural characteristic. We use [5] here. In the tables there (appendices A...) the representations of dimension 48 (except for the natural representation of the classical groups) which appear are the following ones :

- (1) case A_2 , $p \neq 7$
- (2) case A_6 , $p \neq 7$
- (3) case B_3 , $p \neq 7$
- (4) case B_4 , $p = 2$
- (5) case C_3 , $p = 2$
- (6) case C_4 , $p \neq 3$
- (7) case D_4 , $p = 2$

Also, it is not taken into account there the type A_1 , which we can rule out separately, as in this case G would have an abelian p -Sylow subgroup which contradicts the fact that the one of $Q_3 \times Q_4 \times Q_5$ is not.

Since $p \neq 2$, the only cases to consider are then (1),(2),(3),(6). In all cases Q_3, Q_4, Q_5 have p -Sylow subgroups of order g^6, g^6, g^{10} . It follows that a p -Sylow subgroup of $Q_3 \times Q_4 \times Q_5$ has order g^{22} . In cases (1),(2),(3),(6) (and $p \neq 2$) G is contained inside $\mathrm{SL}_3(g), \mathrm{SL}_7(g), \Omega_7(g), \mathrm{PSp}_8(g)$, which have p -Sylow subgroups of order g^3, g^{21}, g^9, g^{16} , respectively, a contradiction.

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