RESIDUAL NILPOTENCE FOR GENERALIZATIONS OF PURE BRAID GROUPS

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Abstract. It is known that the pure braid groups are residually torsion-free nilpotent. This property is however widely open for the most obvious generalizations of these groups, like pure Artin groups and like fundamental groups of hyperplane complements (even reflection ones). In this paper we relate this problem to the faithfulness of linear representations, and prove the residual torsion-free nilpotence for a few other groups.

1. Introduction

It has been known for a long time (see [FR1, FR2]) that the pure braid groups are residually nilpotent, meaning that they have 'enough' nilpotent quotients to distinguish their elements, or equivalently that the intersection of their descending central series is trivial. Recall that a group G is called residually $\mathcal{F}$ for $\mathcal{F}$ a class of groups if for all $g \in G \setminus \{1\}$ there exists $\pi : G \twoheadrightarrow Q$ with $Q \in \mathcal{F}$ such that $\pi(g) \neq 1$. It is also known that they have the far stronger property of being residually torsion-free nilpotent. The strongness of this latter assumption is illustrated by the following implications (where 'residually p' corresponds to the class of p-groups).

$$\text{residually free} \Rightarrow \text{residually torsion-free nilpotent} \Rightarrow \text{residually } p \text{ for all } p$$

$$\Rightarrow \text{residually } p \text{ for some } p \Rightarrow \text{residually nilpotent} \Rightarrow \text{residually finite}$$

Pure braid groups are not residually free. The following proof of this fact has been communicated to me several years ago by Luis Paris (note however that the pure braid group on 3 strands $P_3 \simeq F_2 \times \mathbf{Z}$ is residually free; it has been announced this year that P_4 is not residually free, see [CFR]).

Proposition 1.1. *The pure braid group P_n is not residually free for $n \geq 5$.*

Proof. It is sufficient to show that P_5 is not residually free. Letting $\sigma_1, \dots, \sigma_4$ denote the Artin generators of the braid group B_5, P_5 contains the subgroup H generated by $a = \sigma_1^2$, $b = \sigma_2^2$, $c = \sigma_3^2$, $d = \sigma_4^2$. As shown in [DLS] (see also [Col]) this group is a right-angled Artin group, which contains a subgroup $H_0 =< a, b, d >$ isomorphic to $F_2 \times \mathbf{Z}$, which is well-known to be residually free but not fully residually free (see [Ba]).

If we can exhibit $x \in P_5$ such that the subgroup generated by x and H_0 is a free product $\mathbf{Z} * H_0$, then, by a result of [Ba] which states that the free product of two non-trivial groups can be residually free only if the two of them are fully residually free, this proves that P_5 is not residually free.

One can take $x = cbc^{-1}$. Indeed, if $< x, H_0 >$ were not a free product, then it would exist a word with trivial image of the form $cb^{u_1}c^{-1}y_1cb^{u_2}c^{-1}y_2 \dots cb^{u_r}c^{-1}y_r$ with $y_i \in < a, b, d >$, $y_i \neq 1$ for $i < r$, $u_i \neq 0$ for $i \leq r$, and $r \geq 1$. But in a right-angled Artin group generated by a set X of letters, an expression can be reduced if and only if it contains a word of the form $x \dots x^{-1}$ or $x^{-1} \dots x$ with $x \in X$, such that all the letters in $\dots$ commute with x (see e.g. [Se]). From this it is straightforward to check that the former expression cannot be reduced, and this proves the claim. □

The original approach for proving this property of residual torsion-free nilpotence seems to fail for most of the usual generalizations of pure braid groups. Another approach has been used in

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[Mar1, Mar2], using faithful linear representations, thus relating the linearity problem with this one. The main lemma is the following one.

Lemma 1.2. *Let $N \geq 1$, $\mathbf{k}$ a field of characteristic 0 and $A = \mathbf{k}[[h]]$ the ring of formal power series. Then the group $\mathrm{GL}_N^0(A) = \{X \in \mathrm{GL}_N(A) \mid X \equiv 1 \mod h\} = 1 + h\mathrm{Mat}_N(A)$ is residually torsion-free nilpotent.*

Proof. Let $G = \mathrm{GL}_N^0(A)$, $G_r = \{g \in G \mid g \equiv 1 \mod h^r\}$,

$$G^{(r)} = \{X \in \mathrm{GL}_N(\mathbf{k}[h]/h^r) \mid X \equiv 1 \mod h\} = 1 + h\mathrm{Mat}_N(\mathbf{k}[h]/h^r) \subset \mathrm{Mat}_N(\mathbf{k}[h]/h^r)$$

Clearly $G_r \lhd G$ and the natural map $G \to G^{(r)}$ has for kernel G_r, hence G/G_r is isomorphic to a subgroup of $G^{(r)}$. This latter group is clearly nilpotent, as $(1 + h^u x, 1 + h^v y) \equiv 1 + h^{uv}(xy - yx)$ mod h^{uv+1} (where $(a, b) = aba^{-1}b^{-1}$), and torsion-free as $(1 + hx)^n \equiv 1 + nhx \mod h^2$ and $\mathbf{k}$ has characteristic 0. Thus all the G/G_r are torsion-free nilpotent, and since clearly $\bigcap_r G_r = \{1\}$ we get that G is residually torsion-free nilpotent. ☐

Usually, linear representations have their image in such a group when they appear as the monodromy of a flat connection on a *trivial* vector bundle (see [Mar2]). However, we show how to (partly conjecturally) use this approach in situation where this geometric motivation is far less obvious. In particular, we prove the following.

Theorem 1.3. *If B is an Artin group for which the Paris representation is faithful, then its pure subgroup P is residually torsion-free nilpotent.*

So far, this Paris representation, which is a generalization of the Krammer representation of [K], has been shown to be faithful only for the case where W is a finite Coxeter group. By contrast, in the case of the pure braid groups of complex reflexion groups, which are other natural generalization of pure braid groups, and for which a natural and possibly faithful monodromy representation has been constructed in [Mar3], we get the following more modest but unconditional result.

Theorem 1.4. *If B is the braid group of a complex reflection group of type G_{25}, G_{26}, G_{32}, G_{31}, then its pure braid group P is residually torsion-free nilpotent.*

The pure braid groups involved in the latter statement are equivalently described as the fundamental groups of complements of remarkable configurations of hyperplanes : the groups G_{25}, G_{26} are related to the symmetry group of the so-called Hessian configuration of the nine inflection points of nonsingular cubic curves, while G_{32} acts by automorphisms on the configuration of 27 lines on a nonsingular cubic surface. These groups belong to the special case of so-called 'Shephard groups', namely the symmetry groups of regular complex polytopes. The group G_{31}, introduced by H. Maschke in his first paper [Mas], is not a Shephard group, has all its reflections of order 2, and is connected to the theory of hyperelliptic functions. It is the only 'exceptional' reflection group in dimension $n \geq 3$ which cannot be generated by n reflections.

2. Artin groups and Paris representation

2.1. **Preliminaries on Artin groups.** Let S be a finite set. Recall that a *Coxeter matrix* based on S is a matrix $M = (m_{s,t})_{s,t \in S}$ indexed by elements of S such that

- $m_{ss} = 1$ for all $s \in S$
- $m_{st} = m_{ts} \in \{2, 3, \dots, \infty\}$ for all $s, t \in S$, $s \neq t$.

and that the Coxeter system associated to M is the couple (W, S), with W the group presented by $< S \mid \forall s \in S \; s^2 = 1, \forall s, t \in S \;\; (st)^{m_{st}} = 1 >$. Let $\Sigma = \{\sigma_s, s \in S\}$ be a set in natural bijection with S. The *Artin system* associated to M is the pair (B, Σ) where B is the group presented by $< \Sigma \mid \forall s, t \in S \;\; \underbrace{\sigma_s \sigma_t \sigma_s \dots}_{m_{s,t} \text{ terms}} = \underbrace{\sigma_t \sigma_s \sigma_t \dots}_{m_{s,t} \text{ terms}} >$, and called the Artin group associated to M. The Artin monoid B^+ is the monoid with the same presentation. According to [P], the natural monoid morphism $\sigma_s \mapsto \sigma_s$, $B^+ \to B$, is an embedding. There is a natural morphism $B \twoheadrightarrow W$ given by $\sigma_s \mapsto s$, whose kernel is known as the pure Artin group P.

For the sequel we will need a slightly more specialized vocabulary, borrowed from [P]. A Coxeter matrix is said to be *small* if $m_{s,t} \in \{2,3\}$ for all $s \neq t$, and it is called *triangle-free* if there is no triple (s,t,r) in S such that $m_{s,t}$, $m_{t,r}$ and $m_{r,s}$ are all greater than 2.

2.2. **Paris representation.** To a Coxeter system as above is naturally associated a linear representation of W, known as the reflection representation. We briefly recall its construction. Let $\Pi = \{\alpha_s; s \in S\}$ denote a set in natural bijection with S, called the set of simple roots. Let U denote the $\mathbf{R}$-vector space with basis Π, and $<\ ,\ >: U \times U \to \mathbf{R}$ the symmetric bilinear form defined by

$$< \alpha_s, \alpha_t >= \begin{cases} -2\cos\left(\frac{\pi}{m_{st}}\right) & \text{if} \quad m_{st} < \infty \\ -2 & \text{otherwise} \end{cases}$$

In particular $< \alpha_s, \alpha_s >= 2$. There is a faithful representation $W \to \mathrm{GL}(U)$ defined by $s(x) = x - < \alpha_s, x > \alpha_s$ for $x \in U$, $s \in S$, which preserves the bilinear form $<\ ,\ >$. Let $\Phi = \{w\alpha_s; s \in S, w \in W\}$ be the root system associated to W, $\Phi^+ = \{\sum_{s \in S} \lambda_s \alpha_s \in \Phi; \forall s \in S \ \ \lambda_s \geq 0\}$, and $\Phi^- = -\Phi^+$. We let ℓ denote the length function on W (resp. B^+) with respect to S (resp. Σ). The *depth* of $\beta \in \Phi^+$ is

$$dp(\beta) = \min\{m \in \mathbf{N} \mid \exists w \in W \ \ w.\beta \in \Phi^- \text{ and } \ell(w) = m\}.$$

We have (see [P] lemma 2.5)

$$dp(\beta) = \min\{m \in \mathbf{N} \mid \exists w \in W, s \in S \ \ \beta = w^{-1}.\alpha_s \text{ and } \ell(w) + 1 = m\}.$$

When $s \in S$ and $\beta \in \Phi^+ \setminus \{\alpha_s\}$, we have

$$dp(s.\beta) = \begin{cases} dp(\beta) - 1 & \text{if} \quad \langle \alpha_s, \beta \rangle > 0 \\ dp(\beta) & \text{if} \quad \langle \alpha_s, \beta \rangle = 0 \\ dp(\beta) + 1 & \text{if} \quad \langle \alpha_s, \beta \rangle < 0 \end{cases}$$

In [P], polynomials $T(s,\beta) \in \mathbf{Q}[y]$ are defined for $s \in S$ and $\beta \in \Phi^+$. They are constructed by induction on $dp(\beta)$, by the following formulas. When $dp(\beta) = 1$, that is $\beta = \alpha_t$ for some $t \in S$, then

$$\begin{array}{llcll} \text{(D1)} & T(s,\alpha_t) & = & y^2 & \text{if } t = s \\ \text{(D2)} & T(s,\alpha_t) & = & 0 & \text{if } t \neq s \end{array}$$

When $dp(\beta) \geq 2$, then there exists $t \in S$ such that $dp(t.\beta) = dp(\beta) - 1$, and we necessarily have $b = \langle \alpha_t, \beta \rangle > 0$. In case $\langle \alpha_s, \beta \rangle > 0$, we have

$$\text{(D3)} \quad T(s,\beta) = y^{dp(\beta)}(y-1);$$

in case $\langle \alpha_s, \beta \rangle = 0$, we have

$$\begin{array}{llcll} \text{(D4)} & T(s,\beta) & = & yT(s,\beta - b\alpha_t) & \text{if } \langle \alpha_s, \alpha_t \rangle = 0 \\ \text{(D5)} & T(s,\beta) & = & (y-1)T(s,\beta - b\alpha_t) + yT(t,\beta - b\alpha_s - b\alpha_t) & \text{if } \langle \alpha_s, \alpha_t \rangle = -1; \end{array}$$

and, in case $\langle \alpha_s, \beta \rangle = -a < 0$, we have

$$\begin{array}{llcll} \text{(D6)} & T(s,\beta) & = & yT(s,\beta - b\alpha_t) & \text{if } \langle \alpha_s, \alpha_t \rangle = 0 \\ \text{(D7)} & T(s,\beta) & = & (y-1)T(s,\beta - b\alpha_t) + yT(t,\beta - (b-a)\alpha_s - b\alpha_t) & \text{if } \langle \alpha_s, \alpha_t \rangle = -1 \text{ and } b > a \\ \text{(D8)} & T(s,\beta) & = & T(t,\beta - b\alpha_t) + (y-1)T(s,\beta - b\alpha_t) & \text{if } \langle \alpha_s, \alpha_t \rangle = -1 \text{ and } b = a \\ \text{(D9)} & T(s,\beta) & = & yT(s,\beta - b\alpha_t) + T(t,\beta - b\alpha_t) + y^{dp(\beta)-1}(1-y) & \text{if } \langle \alpha_s, \alpha_t \rangle = -1 \text{ and } b < a. \end{array}$$

Now introduce $\mathcal{E} = \{e_\beta; \beta \in \Phi^+\}$ a set in natural bijection with Φ^+, and let V denote the free $\mathbf{Q}[x,y,x^{-1},y^{-1}]$-module with basis $\mathcal{E}$. For $s \in S$, one defines a linear map $\varphi_s : V \to V$ by

$$\begin{array}{lcll} \varphi_s(e_\beta) & = & 0 & \text{if } \beta = \alpha_s \\ & & e_\beta & \text{if } \langle \alpha_s, \beta \rangle = 0 \\ & & ye_{\beta - a\alpha_s} & \text{if } \langle \alpha_s, \beta \rangle = a > 0 \text{ and } \beta \neq \alpha_s \\ & & (1-y)e_\beta + e_{\beta + a\alpha_s} & \text{if } \langle \alpha_s, \beta \rangle = -a < 0 \end{array}$$

We have $\varphi_s\varphi_t = \varphi_t\varphi_s$ if $m_{s,t} = 2$, $\varphi_s\varphi_t\varphi_s = \varphi_t\varphi_s\varphi_t$ if $m_{s,t} = 3$. Now the Paris representation $\Psi : B \to \mathrm{GL}(V)$ is defined by $\Psi : \sigma_s \mapsto \psi_s$, with

$$\psi_s(e_\beta) = \varphi_s(e_\beta) + xT(s,\beta)e_{\alpha_s}.$$

2.3. **Reduction modulo h.** We embed $\mathbf{Q}[y]$ inside $\mathbf{Q}[[h]]$ under $y \mapsto e^h$ and consider congruences $\equiv$ modulo h. Using the formulas of [P], we deduce the main technical step of our proof.

Proposition 2.1. *Let $s \in S$ and $\beta \in \Phi^+$. Then $T(s,\beta) \equiv 1$ if $\beta = \alpha_s$ and $T(s,\beta) \equiv 0$ otherwise.*

Proof. The case $dp(\beta) = 1$ is a consequence of (D1),(D2), as $y \equiv 1 \mod h$. We thus browse through the various cases when $dp(\beta) \geq 2$, and use induction on the depth. As in the definition of the polynomials, let $t \in S$ such that $dp(\gamma) = dp(\beta) - 1$ for $\gamma = t.\beta$, and recall that necessarily $b = \langle \alpha_t, \beta\rangle > 0$. In case $\langle \alpha_s, \beta\rangle > 0$ then (D3) implies $T(s,\beta) \equiv 0$. If $\langle \alpha_s, \beta\rangle = 0$, we have several subcases. If $\langle \alpha_s, \alpha_t\rangle = 0$, then (D4) implies $T(s,\beta) \equiv T(s,\gamma)$ with $\gamma = \beta - b\alpha_t = t.\beta$ hence $dp(\gamma) < dp(\beta)$ and $T(s,\gamma) \equiv 0$ by induction, unless $\gamma = \alpha_s$, that is $\alpha_s = \beta - b\alpha_t$, hence taking the scalar product by α_s we would get $2 = 0$, a contradiction. Otherwise, we have $\langle \alpha_s, \alpha_t\rangle = -1$. In that case, (D5) implies $T(s,\beta) \equiv T(t, \beta - b\alpha_s - b\alpha_t)$. Note that $\beta - b\alpha_t = \gamma = t.\beta$, $\langle \alpha_s, \gamma\rangle = 0 - b\langle \alpha_t, \alpha_s\rangle = b$ and $s.\gamma = \gamma - b\alpha_s$. Thus $T(s,\beta) \equiv T(t, st.\beta)$. Now $\langle \alpha_s, \gamma\rangle = b > 0$ hence $dp(s.\gamma) = dp(\gamma) - 1 < dp(\beta)$, unless $\gamma = \alpha_s$; but the case $\gamma = \alpha_s$ cannot occur here, as it would imply

$$2 = \langle \alpha_s, \alpha_s\rangle = \langle \alpha_s, \gamma\rangle = \langle \alpha_s, t.\beta\rangle = \langle \alpha_s, \beta - b\alpha_t\rangle = -b\langle \alpha_s, \alpha_t\rangle = b$$

hence $\alpha_s = \gamma = t.\beta = \beta - 2\alpha_t$, whence $-1 = \langle \alpha_s, \alpha_t\rangle = \langle \beta, \alpha_t\rangle - 2\langle \alpha_t, \alpha_t\rangle = b - 4 = -2$, a contradiction.

Finally, $T(t, st.\beta) \equiv 0$ by induction unless $\beta - b\alpha_s - b\alpha_t = \alpha_t$, in which case scalar product by α_t leads to the contradiction $2 = 0$.

The last case is when $\langle \alpha_s, \beta\rangle = -a < 0$, which is subdivided in 4 subcases. Either $\langle \alpha_s, \alpha_t\rangle = 0$, and then (D6) implies $T(s,\beta) \equiv T(s, \beta - b\alpha_t) \equiv 0$, as $\beta - b\alpha_t = \alpha_s$ cannot occur (scalar product with α_s yields $2 = -a < 0$). Or $\langle \alpha_s, \alpha_t\rangle = -1$ and $b > a$, then (D7) implies $T(s,\beta) \equiv T(t, \beta - (b-a)\alpha_s - b\alpha_t) \equiv 0$ unless $\alpha_t = \beta - (b-a)\alpha_s - b\alpha_t$, which cannot occur for the same reason as before (take the scalar product with α_t). Or, $\langle \alpha_s, \alpha_t\rangle = -1$ and $b = a$, in which case (D8) implies $T(s,\beta) \equiv T(t, \beta - b\alpha_t) \equiv 0$, as $\alpha_t \neq \beta - b\alpha_t$ (take the scalar product with α_t). Finally, the last subcase is $\langle \alpha_s, \alpha_t\rangle = -1$ and $b < a$, then (D9) implies $T(s,\beta) \equiv T(s, \beta - b\alpha_t) + T(t, \beta - b\alpha_t) \equiv 0$, unless $\alpha_s = \beta - b\alpha_t$, which leads to the contradiction $2 = b - a < 0$ under $\langle \alpha_s, \cdot\rangle$, or $\alpha_t = \beta - b\alpha_t$, which leads to the contradiction $2 = -b < 0$ under $\langle \alpha_t, \cdot\rangle$. ☐

We now embed $\mathbf{Q}[x^{\pm 1}, y^{\pm 1}]$ into $\mathbf{Q}(\sqrt{2})[[h]]$ under $y \mapsto e^h$, $x \mapsto e^{\sqrt{2}h}$ (any other irrational than $\sqrt{2}$ would also do), and define $\tilde{V} = V \otimes_\iota \mathbf{Q}(\sqrt{2})[[h]]$ where ι is the chosen embedding; that is, $\tilde{V}$ is the free $\mathbf{Q}(\sqrt{2})[[h]]$-module with basis $\mathcal{E}$, and clearly $V \subset \tilde{V}$. We similarly introduce the $\mathbf{Q}(\sqrt{2})$-vector space V_0 with basis $\mathcal{E}$. One has $\mathrm{GL}(V) \subset \mathrm{GL}(\tilde{V})$, and a reduction morphism $\mathrm{End}(\tilde{V}) \to \mathrm{End}(V_0)$. Composing both we get elements $\overline{\psi_s}, \overline{\varphi_s} \in \mathrm{End}(V_0)$ associated to the $\psi_s \in \mathrm{GL}(V)$, $\varphi_s \in \mathrm{End}(V)$. Because $x \equiv 1 \mod h$ and because of proposition 2.1 one gets from the definition of ψ_s that

$$\begin{array}{rcll} \overline{\psi}_s(e_\beta) & = & \overline{\varphi_s}(e_\beta) & \text{if } \beta \neq \alpha_s \\ & = & \overline{\varphi_s}(e_{\alpha_s}) + e_{\alpha_s} = e_{\alpha_s} & \text{if } \beta = \alpha_s. \end{array}$$

We denote $(w,\beta) \mapsto w \star \beta$ the natural action of W on Φ^+, that is $w \star \beta = \beta$ if $w.\beta \in \Phi^+$, $w \star \beta = -\beta \in \Phi^+$ if $w.\beta \in \Phi^-$. The previous equalities imply

$$\forall s \in S \quad \forall \beta \in \Phi^+ \quad \overline{\psi_s}(e_\beta) = e_{s\star\beta}$$

From this we deduce the following.

Proposition 2.2. *For all $g \in P$, $\overline{\Psi(g)} = \mathrm{Id}_{V_0}$.*

Proof. Recall that P is defined as $\mathrm{Ker}(\pi : B \twoheadrightarrow W)$. From $\overline{\psi_s}(e_\beta) = e_{s\star\beta}$ one gets $\overline{\Psi(g)}(e_\beta) = e_{\pi(g)\star\beta}$ for all $g \in B$ and the conclusion. ☐

As a consequence $\Psi(P) \subset \{\varphi \in \mathrm{GL}(V) \mid \overline{\varphi} = \mathrm{Id}_{V_0}\}$. The group $\{\varphi \in \mathrm{GL}(V) \mid \overline{\varphi} = \mathrm{Id}_{V_0}\}$ is a subgroup of $G = \{\varphi \in \mathrm{GL}(\tilde{V}) \mid \overline{\varphi} = \mathrm{Id}_{V_0}\}$, which we now prove to be residually torsion-free nilpotent. We adapt the argument of lemma 1.2 to the infinite-dimensional case. Let $\mathbf{k} = \mathbf{Q}(\sqrt{2})$. The canonical projection $\mathbf{k}[[h]] \twoheadrightarrow \mathbf{k}[h]/h^r$ extends to a morphism $\pi_r : \mathrm{End}(\tilde{V}) \to \mathrm{End}(V_0) \otimes_{\mathbf{k}} \mathbf{k}[h]/h^r$ with clearly $\pi_1(\varphi) = \overline{\varphi}$. Let $G_r = \{\varphi \in \mathrm{GL}(\tilde{V}) \mid \pi_r(\varphi) = \mathrm{Id}\}$. Then G/G_r is identified to $\pi_r(G)$ which is a subgroup of $\{\mathrm{Id}_{V_0} + hu \mid u \in \mathrm{End}(V_0) \otimes_{\mathbf{k}} \mathbf{k}[h]/h^r\}$, which is clearly torsion-free and nilpotent. Since $\bigcap_r G_r = \{1\}$ this proves the residual torsion-free nilpotence of G and theorem 1.3.

3. Braid groups of complex reflection groups

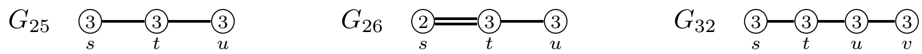
A pseudo-reflection in $\mathbf{C}^n$ is an endomorphism which fixes an hyperplane. For W a finite subgroup of $\mathrm{GL}_n(\mathbf{C})$ generated by pseudo-reflections (so-called complex reflection group), we have a *reflection arrangement* $\mathcal{A} = \{\mathrm{Ker}(s-1) | s \in \mathcal{R}\}$, where $\mathcal{R}$ is the set of reflections of W. Letting X denote the hyperplane complement $X = \mathbf{C}^n \setminus \bigcup \mathcal{A}$, the fundamental groups $P = \pi_1(X)$ and $B = \pi_1(X/W)$ are called the pure braid group and braid group associated to W. The case of spherical type Artin groups corresponds to the case where W is a finite Coxeter group.

It is conjectured that P is always residually torsion-free nilpotent. For this one can assume that W is irreducible. According to [ST], such a W belongs either to an infinite series $G(de, e, n)$ depending on three integer parameters d, e, n, or to a finite set of 34 exceptions, denoted $G_4, \dots, G_{37}$. The fiber-type argument of [FR1, FR2] to prove the residual torsion-free nilpotence only works for the groups $G(d, 1, n)$, and when $n = 2$.

For the case of W a finite Coxeter group, we used the Krammer representation to prove that P is residually torsion-free nilpotent in [Mar1, Mar2]. The exceptional groups of rank $n > 2$ which are not Coxeter groups are the 9 groups G_{24}, G_{25}, G_{26}, G_{27}, G_{29}, G_{31}, G_{32}, G_{33}, G_{34}.

We show here that this argument can be adjusted to prove the residual torsion-free nilpotence for a few of them.

We begin with the Shephard groups G_{25}, G_{26}, G_{32}. The Coxeter-like diagrams of these groups are the following ones.



It is known (see [BMR]) that removing the conditions on the order of the generators gives a (diagrammatic) presentation of the corresponding braid group. In particular, these have for braid groups the Artin groups of Coxeter type A_3, B_3 and A_4, respectively.

We recall a matrix expression of the Krammer representation for B of Coxeter type A_{n-1}, namely for the classical braid group on n strands. Letting $\sigma_1, \dots, \sigma_{n-1}$ denote its Artin generators with relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|j - i| \geq 2$, $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, their action on a specific basis x_{ij} $(1 \leq i < j \leq n)$ is given by the following formulas (see [K])

$$\left\{ \begin{array}{ll} \sigma_k x_{k,k+1} = tq^2 x_{k,k+1} & \\ \sigma_k x_{i,k} = (1-q)x_{i,k} + qx_{i,k+1} & i < k \\ \sigma_k x_{i,k+1} = x_{i,k} + tq^{k-i+1}(q-1)x_{k,k+1} & i < k \\ \sigma_k x_{k,j} = tq(q-1)x_{k,k+1} + qx_{k+1,j} & k+1 < j \\ \sigma_k x_{k+1,j} = x_{k,j} + (1-q)x_{k+1,j} & k+1 < j \\ \sigma_k x_{i,j} = x_{i,j} & i < j < k \text{ or } k+1 < i < j \\ \sigma_k x_{i,j} = x_{i,j} + tq^{k-i}(q-1)^2 x_{k,k+1} & i < k < k+1 < j \end{array} \right.$$

where t and q denote algebraically independent parameters. We embed the field $\mathbf{Q}(q,t)$ of rational fractions in q, t into $K = \mathbf{C}((h))$ by $q \mapsto -\zeta_3 e^h$ and $t \mapsto e^{\sqrt{2}h}$, where ζ_3 denotes a primitive 3-root of 1. We then check by an easy calculation that $\sigma_k^3 \equiv 1$ modulo h. Since the quotients of the braid group on n strands by the relations $\sigma_k^3 = 1$ are, for $n = 3, 4, 5$, the Shephard group of types G_4, G_{25} and G_{32}, respectively, it follows that the pure braid groups of these types embed in $\mathrm{GL}_N^0(A)$ with $N = n(n-1)/2$ and $\mathbf{k} = \mathbf{C}$, and this proves their residual torsion-free nilpotence by lemma 1.2.

We now turn to type G_{26}. Types G_{25} and G_{26} are symmetry groups of regular complex polytopes which are known to be closely connected (for instance they both appear in the study of the Hessian configuration, see e.g. [Cox] §12.4 and [OT] example 6.30). The hyperplane arrangement of type G_{26} contains the 12 hyperplanes of type G_{25} plus 9 additional ones. The natural inclusion induces morphisms between the corresponding pure braid groups, which cannot be injective, since a loop around one of the extra hyperplanes is non trivial in type G_{26}. However we will prove the following, which proves the residual torsion-free nilpotence in type G_{26}.

Proposition 3.1. *The pure braid group of type G_{26} embeds into the pure braid group of type G_{25}.*

More precisely, letting B_i, P_i, W_i denote the braid group, pure braid group and pseudo-reflection group of type G_i, respectively, we construct morphisms $B_{26} \hookrightarrow B_{25}$ and $W_{26} \twoheadrightarrow W_{25}$ such that the following diagram commutes, where the vertical arrows are the natural projections.

$$\begin{array}{ccc} B_{25} & \hookleftarrow & B_{26} \\ \downarrow & & \downarrow \\ W_{25} & \twoheadleftarrow & W_{26} \end{array}$$

Both horizontal morphisms are given by the formula $(s,t,u) \mapsto ((tu)^3, s, t)$, where s,t,u denote the generators of the corresponding groups according to the above diagrams. The morphism between the pseudo-reflection groups is surjective because it is a retraction of an embedding $W_{25} \hookrightarrow W_{26}$ mapping (s,t,u) to $(t,u,t^{sut^{-1}u})$. The kernel of this projection is the subgroup of order 2 in the center of W_{26} (which has order 6).

We now consider the morphism between braid groups and prove that it is injective. First recall that the braid group of type G_{26} can be identified with the Artin group of type B_3. On the other hand, Artin groups of type B_n are isomorphic to the semidirect product of the Artin group of type A_{n-1}, that we denote $\mathcal{B}_n$ to avoid confusions, with a free group F_n on n generators $g_1, \dots, g_n$, where the action (so-called 'Artin action') is given (on the left) by

$$\sigma_i : \begin{cases} g_i & \mapsto \quad g_{i+1} \\ g_{i+1} & \mapsto \quad g_{i+1}^{-1} g_i g_{i+1} \\ g_j & \mapsto \quad g_j \qquad\qquad \text{if } j \notin \{i, i+1\} \end{cases}$$

If $\tau, \sigma_1, \dots, \sigma_{n-1}$ are the standard generators of the Artin group of type B_n, with $\tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau$, $\tau\sigma_i = \sigma_i\tau$ for $i > 1$, and usual braid relations between the σ_i, then this isomorphism is given by $\tau \mapsto g_1$, $\sigma_i \mapsto \sigma_i$ (see [CP] prop. 2.1 (2) for more details). Finally, there exists an embedding of this semidirect product into the Artin group $\mathcal{B}_{n+1}$ of type A_n which satisfies $g_1 \mapsto (\sigma_2 \dots \sigma_n)^n$, and $\sigma_i \mapsto \sigma_i$ $(i \leq n-1)$. By composing both, we get an embedding which makes the square commute. This proves proposition 3.1.

This embedding of type B_n into type A_n, different from the more standard one $\tau \mapsto \sigma_1^2, \sigma_i \mapsto \sigma_{i+1}$, has been considered in [L]. The algebraic proof given there being somewhat sketchy, we provide the details here. This embedding comes from the following construction.

Consider the (faithful) Artin action as a morphism $\mathcal{B}_{n+1} \to \mathrm{Aut}(F_{n+1})$, and the free subgroup $F_n = \langle g_1, \dots, g_n \rangle$ of F_{n+1}. The action of $\mathcal{B}_{n+1}$ preserves the product $g_1 g_2 \dots g_{n+1}$, and there is a natural retraction $F_{n+1} \twoheadrightarrow F_n$ which sends g_{n+1} to $(g_1 \dots g_n)^{-1}$. This induces a map $\Psi : \mathcal{B}_{n+1} \to \mathrm{Aut}(F_n)$, whose kernel is the center of $\mathcal{B}_{n+1}$ by a theorem of Magnus (see [Mag]). We claim that its image contains the group $\mathrm{Inn}(F_n)$ of inner automorphisms of F_n, which is naturally isomorphic to F_n.

Indeed, it is straightforward to check that $b_1 = (\sigma_2 \dots \sigma_n)^n$ is mapped to $\mathrm{Ad}(g_1) = x \mapsto g_1 x g_1^{-1}$. Defining $b_{i+1} = \sigma_i b_i \sigma_i^{-1}$, we get that b_i is mapped to $\mathrm{Ad}(g_i)$. In particular the subgroup $\mathcal{F}_n = \langle b_1, \dots, b_n \rangle$ of $\mathcal{B}_{n+1}$ is free and there is a natural isomorphism $\varphi : b_i \mapsto g_i$ to F_n characterized by the property $b.g = \mathrm{Ad}(\varphi(b))(g)$ for all $g \in F_n$ and $b \in \mathcal{F}_n$, that is $\mathrm{Ad}(\varphi(b)) = \Psi(b)$ for all $b \in \mathcal{F}_n$.

Now, let $\mathcal{B}_n \subset \mathcal{B}_{n+1}$ be generated by $\sigma_i, i \leq n-1$. Its action on F_n is the usual Artin action recalled above. For $\sigma \in \mathcal{B}_n$ and $b \in \mathcal{F}_n$ we know that

$$\forall x \in F_n \quad \sigma b \sigma^{-1}.x = \sigma.\left(\varphi(b)(\sigma^{-1}.x)\varphi(b)^{-1}\right) = (\sigma.\varphi(b))x(\sigma.\varphi(b))^{-1}$$

that is $\sigma b \sigma^{-1}$ is mapped to $\mathrm{Ad}(\sigma.\varphi(b))$ in $\mathrm{Aut}(F_n)$, hence $\sigma b \sigma^{-1}$ and $\varphi^{-1}(\sigma.\varphi(b)) \in \mathcal{F}_n$ have the same image under Ψ. Since the kernel of Ψ is $Z(\mathcal{B}_{n+1})$, this proves that they may differ only by an element of the center $Z(\mathcal{B}_{n+1})$ of $\mathcal{B}_{n+1}$. On the other hand, $\varphi : \mathcal{F}_n \to F_n$ commutes with the maps $F_n \to \mathbf{Z}$ and $\eta : \mathcal{F}_n \to \mathbf{Z}$ which map every generator to 1. Likewise, the Artin action commutes with $F_n \to \mathbf{Z}$ hence $\eta(\varphi^{-1}(\sigma.\varphi(b))) = \eta(b)$. We denote $\ell : \mathcal{B}_{n+1} \to \mathbf{Z}$ the abelianization map. We have $\ell(b_i) = n(n-1)$ for all i, hence $\ell(b) = n(n-1)\eta(b)$ for all $b \in \mathcal{F}_n$. Since $\ell(b) = \ell(\sigma b \sigma^{-1})$ it follows that $\sigma b \sigma^{-1}$ and $\varphi^{-1}(\sigma.\varphi(b)) \in \mathcal{F}_n$ differ by an element in $Z(\mathcal{B}_{n+1}) \cap (\mathcal{B}_{n+1}, \mathcal{B}_{n+1})$, where $(\mathcal{B}_{n+1}, \mathcal{B}_{n+1})$ denotes the commutators subgroup. But $Z(\mathcal{B}_{n+1})$ is generated by $(\sigma_1 \dots \sigma_n)^{n+1} \notin (\mathcal{B}_{n+1}, \mathcal{B}_{n+1})$ hence $\sigma b \sigma^{-1} = \varphi^{-1}(\sigma.\varphi(b)) \in \mathcal{F}_n$.

In particular $\mathcal{F}_n$ is stable under the action by conjugation of $\mathcal{B}_n$, which coincides with the Artin action. This is the embedding $\mathcal{B}_n \ltimes F_n \hookrightarrow \mathcal{B}_{n+1}$ that is needed to make the square commute. It remains to prove that we indeed have a semidirect product, namely that $\mathcal{B}_n \cap \mathcal{F}_n = \{1\}$. First notice that $\mathcal{F}_n$ is mapped to $\mathrm{Inn}(F_n)$ and recall that the outer Artin action $\mathcal{B}_n \to \mathrm{Out}(F_n)$ has for kernel $Z(\mathcal{B}_n)$, hence $\mathcal{F}_n \cap \mathcal{B}_n \subset Z(\mathcal{B}_n)$. Then $x \in \mathcal{F}_n \cap \mathcal{B}_n$ can be written $x = z^k$ for some $k \in \mathbf{Z}$ with $z = (\sigma_1 \dots \sigma_{n-1})^n$. It is classical and easy to check that the action of z on F_n is given by $\mathrm{Ad}((g_1 \dots g_n)^{-1})$, hence $\varphi(z^k) = \varphi((b_1 \dots b_n)^{-k})$ and $x = z^k = (b_1 \dots b_n)^{-k}$. Thus $\ell(x) = kn(n-1) = -kn^2(n-1)$ hence $k = 0$ and $x = 1$.

This concludes the case of G_{26}. The case of G_{31} is a consequence of the lifting of Springer's theory of 'regular elements' for complex reflection groups to their associated braid group. By Springer theory (see [Sp]), W_{31} appears as the centralizer of a regular element c of order 4 in W_{37}, which is the Coxeter group of type E_8, and, as a consequence of [Be, thm. 12.5 (iii)], B_{31} can be identified with the centralizer of a lift $\tilde{c} \in B_{37}$ of c, in such a way that the natural diagram

$$\begin{array}{ccc} B_{31} & \hookrightarrow & B_{37} \\ \downarrow & & \downarrow \\ W_{31} & \hookrightarrow & W_{37} \end{array}$$

commutes. This embedding $B_{31} \hookrightarrow B_{37}$ is explicitly described in [DMM], to which we refer for more details. By commutation of the above diagram it induces an embedding $P_{31} \hookrightarrow P_{37}$. Since P_{37} is known to be residually torsion-free nilpotent by [Mar1, Mar2], this concludes the proof of theorem 1.4.

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